Ancient Indian Mathematics
AND
Vedha

L. V. GURJAR

ANCIENT INDIAN MATHEMATICS
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VEDHA

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PREFACE

The achievements of the Ancient Indians in the field of Mathematics and Astronomy form the most wonderful chapters in the history of civilization. The present work, it is hoped, would give in a nutshell the salient contributions made by Indians, to the development of the subjects from Vedic Age to the end of the seventeenth century.

The present work was finished as early as January 1942 just after my visit to the observatories of Ujjain, Jaipur, Delhi and Benares. But due to various difficulties it could not be brought out earlier.

I cannot close this small preface without thanking Mr. D. T. Adhatrao, D. M. E. Assistant Lecturer Poona Engineering College and Mr. V. G. Kelkar for drawing the figures in this book.

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Siddharth College
Bombay
1st April 1947.

L. V. GURJAR

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ANCIENT INDIAN MATHEMATICS

CHAPTER I

INTRODUCTION

History of Mathematics will have to undergo a considerable change due to the recent researches in the history of Ancient Indian Mathematics. It has been found and acknowledged on all hands that the Science of Mathematics was developed to a great perfection and it is the early researches of the Hindus that wielded a vast influence in the forward world march of the science. It is proposed here to place before the reader the salient features of the early developments in this branch of science.

The contributions that the ancient Hindus made to Mathematics can fairly be divided into three periods (a) the Vedic Age (b) the Pre-mediæval period and (c) the mediæval period.

Vedas and Vedāṅgas

The earliest available works are the Vedas. They are the hymns of praise of the Supreme and their highly poetic nature clearly indicate the high state of civilization that flourished from the shores of the Sindhu to the heights of the Himalayas. There is a considerable controversy regarding the age of the Vedas. Some claim that they are the compositions as old as 3000 B. C. but without entering into the details of controversial nature we can take the general estimate of the nature of things and can place the Vedāṅgas—the compositions of the latter part of the Vedic age—in a period extending from

1500 B. C. to 750 B. C. (Winternitz' History of Indian Literature Page 310).

Of course it is clear from the results derived and the developments recorded in a few sūtras in 'Kalpa' and 'Jyotiṣa'—two of the Vedāṅgas, that much of the earlier works must have been lost in antiquity. Still the knowledge that has been handed down or rather transferred vocally for generations together, bears a strong testimony that the Hindu—the inhabitant of the bank of the Sindhu—had made enormous strides in the field of Mathematics. This high level of intellectual attainments was maintained in the mediæval period inspite of foreign invasions, the internecine wars, the vastness of the country and the greatest handicaps such as imperfect channels of communication; with all these difficulties however it can legitimately be said that India lead the whole world in the field of Mathematics as far as the beginning of the 17th century.

Bias of the Western Scholar

With the introduction of the British Raj in India, the western scholars actuated by the thirst of knowledge, swooped upon this abundant treasure of mathematical knowledge almost like carrion birds. They brought out the translations of various works at times with comments, at times with criticisms and at times by bare literal renderings. But it must be borne in mind that their comprehensions, presumptions, explanations, interpretations and inferences cannot be taken word by word. The reasons for this are the following:—

(i) They are alien to the language in which the whole mass of work is found; they often find the language obscure and mystic and moreover it frequently appears that the works are translated with the help of Sanskrit

Pundits—scholars of the language—who knew little of the subject discussed.

(ii) The researches appear in the verse form adapted to the aid of memory, though suitable to the Indian mind, presents a strong difficulty to the westerners, at times rendering the parts unintelligible.

(iii) The whole of Indian atmosphere which forms the background of all developments is foreign to them.

(iv) The western Scholar being foreign to the language is apt to study first the prominent works of Indian Mathematics—almost all of them seem to have begun their research by starting reading *Līlāvatī* by Bhaskaracarya, Brahmasphuta–Siddhānta by Brahmagupta, Ganitasārasangraha by·Mahāvīr &c.–the much developed and wellknown works of the mediæval period. Then they attempt at understanding the developments in Mathematics of the vedic age found in Śulva-Sūtra. The result is that they fail to find the same kind of precision and accuracy in the method of expounding the subject and they therefore conclude that these ancient works, must have been interpolations done at a later date.

(v) They question the originality of the Indian Science by trying to establish the influence of the Greek, Arabian or Egyptian Science over the Indian, without any direct evidence and they assert that these works are influenced by the developments of the Science in the Mediterranean countries. Thus they study the old works rather in a comparative method, through the eye of a biased scholar and not through the impartial eye of a student. Very few of them like Bailly, Sir William Jones, Hankel have pointed out the independence of the growth of the Indian Science.

The question of date arises only in the case of works of the Vedic Age and those of the mediæval period. The age of the Vedas and the Vedāṅgas is a matter of much dispute, but as has been already referred to, we can take the general estimate and can place the Vendāṅgas in a period extending from 1500 B. C. to 750 B. C.

Vedāṅgas

The developments of this age are recorded in the vedāṅgas that are auxiliary to the Vedas. They are designed to aid in the correct pronunciation and interpretations of the texts and the right employment of the mantras or chants in ceremonials. Vedāṅgas are six in number.

शिक्षा कल्पो व्याकरणं निरुक्तं छन्दसां चयः ।
ज्योतिषामयनं चैव वेदांगानि षडेव तु ॥

(i) शिक्षा—the science of proper articulation and pronunciation–Phonetics

(ii) छंदस्—the science of prosody i. e. Metronomy

(iii) व्याकरण—Grammar.

(iv) निरुक्त—Etymology.

(v) ज्योतिष—Astronomy.

(vi) कल्प—Rules for the rituals and the ceremonials. Thus in Vedāṅgas we find the foundations of almost all the essential branches of human knowledge. The last two Vedāṅgas—Jyotiṣ and Kalpa contain the developments in Mathematics of the age.

Of these, the Jyotiṣ helped the ancient Indians in determining the auspicious day, hour and time for performing the sacrifices. It is found in two rescensions—the Ārca–Jyotiṣ and the Yājus-Jyotiṣ. One belongs to Ṛgveda and the other to the Yajurveda. The contents

of both are more or less the same but they differ in number of verses contained in them due perhaps to two different preceptors. In all there are fortynine verses and they give a systematic development of Astronomy together with the Mathematical formulæ required for calculating the time and the positions of the sun and the moon in various *nakṣatras* or the parts of the zodiac.

Sulva Sutra

The other mathematical developments are recorded in the works known as Śulva–Sūtras. Sulva–Sūtras are the supplements of the Kalpa the last of the Vedāṅgas. They give systematically arranged descriptions of sacrificial rites which were practised in Ancient India in Vedic age. They deal with the measurement and construction of different Vedis or altars. The word 'Śulva' means the sacrificial rite and hence 'Śulva–Sūtras' means the rules regarding the 'sacrificial rites'. It seems that later on the word Sulva came to mean the cord employed in measuring the altars. It must be borne in mind that word rajju (रज्जु) is used in the Sūtras to mean the cord and not the word Śulva. Thus the word Śulva–Sūtra must be interpreted as the science dealing with the rules of the sacrificial rites and not to mean 'the rules of the cord' as the western scholars have done. In Śulva-Sūtras are treasured a wealth of geometrical and arithmetical results. They must mainly be considered as works on Engineering with definite instructions for building altars for which all knowledge about numbers has been used to advantage, just as in modern times all available Mathematics is made use of in Engineering. The works on Engineering are not expected to give the proofs of the mathematical results or to present the mathematical rigour as their aim is simply to make use of the formulæ and not to expound the methods by which it is obtained.

Premediæval and mediæval works

In the pre-mediæval period there are two celebrated works (i) the Bakṣāli Manuscript and the Sūrya-Siddhānta. There is a good deal of controversy regarding the date of these works and it is really very difficult to establish the actual date of their composition. Certain circumstantial evidence together with the internal one however, enables us to fix the upper and lower limits for them. Of these Sūrya-Siddhānta is a work on Astronomy, while Bakṣāli Manuscript is purely a work on Arithmetic and Algebra.

In the mediæval period however—that is from the 4th century to the 17th century—there is no need to enter into the discussion regarding the date of various works as the time when the various mathematical celebrities lived, can be definitely known either by the mention of it in the works themselves or by other circumstantial references.

Of course such controversies are to be entered into with a pious desire to record the gratitude of mankind to the originators of various sciences so that the credit must go to the proper personages irrespective of the country or the region they belong to. In such investigations it is never desirable to bring in patriotic, regional or continental considerations. On the other hand one must be prepared to know the facts as they are. Unfortunately, while dealing with developments in Mathematics of the mediæval period, it would be found that many of the western scholars have tried to show in Indian Mathematics the traces of Diophantus—the Greek Mathematician. Hence it is worth-while to note in a nutshell the notations of Diophantus and those of Hindu Mathematicians so that the reader can judge for himself with certainty as to whether there is any grain of truth regarding the Diophantine influence on Indian Mathematics.

Comparison between Diophantus and Hindu notations

The Hindu Mathematicians have no mark for (+); they made use of abbreviations and initials of the word meaning summation for symbol i.e. यु from युत for summation. The mark for minus is a dot over the quantity to be subtracted. Instead of a mark of multiplication, the factors are written together just as in modern way. A fraction is denoted by writing the divisor under the dividend, without a horizontal line of separation. No equality or inequality signs are used. But this fact is denoted by the initial syllable of a word of that import. The unknown quantities are denoted by the first word of the colours. The sides of an equation are written one below the other. The square of an unknown quantity is marked by adding to the expression the first letter of the word which means square (व for वर्ग=square); in like manner the cube and so on.

Now compare these with Diophantine notations. For additions Diophantus does not make use of any symbol at all; it is expressed by mere juxtaposition. For subtraction, the symbol ↑ might have been used by him. It is doubtful whether Diophantus used it, as various theories are advanced by various scholars about it. The probable symbols for subtraction given by various scholars are

↑, ↑, ⋔, ⋔, ⋔

As nobody is definite about it we cannot ascribe its introduction to Diophantus. For division, Diophantus found that no symbol was necessary as long as the divisor divided the dividend without a remainder. In other cases, the quotient is to be expressed as a fraction, and fractions are explained in different ways according as they

are sub-multiples or not, sometimes by simple accent or two accents or sometimes by $\frown$. Thus

$$\gamma^x = \tfrac{1}{3} \text{ or } \gamma'' = \tfrac{1}{3}\ \angle' = \tfrac{1}{2}\ \widehat{\theta} = \tfrac{1}{9}$$

and so on. So no particular definition was used. In other fractions the numerator and the denominator are written in full, the denominator being written *above the numerator*—exactly the reverse of the Indian or the modern way of writing. In his solution he did not introduce more than one unknown. In fact he did not conceive of solutions containing two or more than two unknowns, and if he comes across such a problem, that problem reduces to the indeterminate simultaneous equation of the first degree. Thus his incapability of conceiving the introduction of more than one unknown, is unnecessarily given prominence by the western scholars and on this score, they assert that he is the originator of the indeterminate equations. He called the unknown ακριθμΘ representing by the final ς, the square by *dynamis*, *Cubos* for cube etc. The sixth power for him is cubo-cubous which the Hindus correctly designate as quadrate cube (cube of the square or square of the cube).

Originality of Indian Mathematics

It is true that the Diophantus lived about two centuries before Āryabhaṭṭa. But the spirit of Diophantus is absent from the works of the Hindus. None of his problems, not a trace of his symbolism, not a bit of his phraseology appear in the Indian algebraical work. It is true that the Indian Mathematics lack the cold logic, the consecutive arrangements and the abstract character of modern Mathematics. But it is the noteworthy feature of the Indian Mathematics of this age that it contains problems clothed in highly poetic language rich with imagination—the very trait, the western Mathematics lacks—the trait which alone, is sufficient to establish the originality of Indian science.

Refutation of Cajori's charge against Hindu Mathematicians of the later age

Another charge that has been levelled against the Mathematicians of this period, is that they did not utilise the results obtained by the composers of the Śulvasūtras. Cajori says* "strange to say, none of the geometrical constructions occur in later Hindu works, the latter ignore Mathematical contents of the Śulvasūtras." The same charge would have been made by Cajori with respect to the contents of Bakṣāli Manuscript had he known that it contained more Algebra than "the arithmetical computation"! Such a charge is made without taking into consideration some of the facts in the proper light.

(1) If we look to the geographical conditions of India we find that it is as vast as a continent.

(2) The sacrifices for which the geometrical constructions of the Śulvasūtras were essential, were no longer performed due to the disturbed conditions of the country throughout this period.

(3) If we study the history of Hindu Mathematics and the lives of various Mathematicians of this age, we find that this subject was being studied for generations after generations in one family and the link was kept up throughout till the dullards or the nonmathematical minded sons of the family left the subject. Again whatever they studied had to be put in the form of a verse and as such, only the eminent composers—or poets of the family came to the forefront and are now known as the mathematical celebrities. As such, the illustrations of the Śulva

* Cajori's History of Mathematics.

theorem (wrongly known as Pythagoras) were given in arithmetical numbers afterwards, and by the time their use was handed down to posterity, the original reference might have been lost. The examples given on mensuration to illustrate the use of Śulva theorem have been given by Bhāskarācārya; it is very likely—if we take into consideration that Bhāskar was a son in the seventh generation of the family of a galaxy of Mathematicians keeping up the family traditions—that by the time Bhāskarācārya made use of them, he might have found the acknowledgement of the importance of the methods of Śulvasūtras unnecessary.

(4) The Geometry they developed, had very little to do with the Astronomy they developed and therefore the geometrical theorems referred to, in Sulvasūtras were given as those under mensuration.

Again if we look to the language of the Śulvasūtras and the terminology of the Mathematics of this age, we find a close connection between them. For instance the term Varga (वर्ग), the meaning in which this term has been used by the Mathematicians of this age is "the square of a number." The origin of this term is found in Śulvasūtras. Note down the passage in Āpastamba.

यावत्प्रमाणा रज्जुस्तावतस्तावतो वर्गान्करोति ।

Translation:—As many measures a cord contains, so many rows it produces.

Thus the square drawn with the side of five 'puruṣas' could be divided into five 'Vargas' each consisting of five small squares each with a side of one

Puruṣa. Another word that establishes this connection is 'Karaṇī' (करणी). 'Karaṇī' meant originally the 'rajjukaraṇī' (रज्जुकरणी) the cord used for measuring of a square altar by the Adhvaryus. To the composers of the Śulvasūtras it was an easy problem to draw a square whose side is an irrational number say $\sqrt{10}$. For

पदं तिर्यङ्मानी त्रिपदापार्श्वमानी तस्याक्ष्णया रज्जुर्दशकरणी ।

$$\text{i. e. } 1^2 + 3^2 = 10$$

The Karaṇī that measured such a side represented an irrational number, and thus the word Karaṇī came to have the meaning a proper surd, just as $\sqrt{3}$, $\sqrt{5}$, $\sqrt{7}$ etc. How the word came to have the meaning cannot be explained in any other way but this, and that establishes the connection between this and the Śulvasūtras. The word Citi (चिति) is also taken from Sulvasūtras.

Hence such charges made by the historian of Mathematics are ill-founded.

Whether Hindus borrowed from Greeks or Greeks borrowed from Hindus

Another charge that has been made against the Indian Mathematics is, to quote Cajori. "The Hindoo gave no definition, no postulates, no axioms, no logical chain of reasoning. His knowledge of mensuration was largely borrowed from the Mediterranian and from China, through *imperfect channels of communications*".*

As regards the first part of the statement by Cajori we have to note down the following points. In the first place, they have to put everything in the verse form to be memorised easily, they wanted to make it as short as possible and as such, they noted down the final results they derived, in more or less the Sūtra form (or formulae). Note that in all cases they followed this method. Whenever a difficult problem was tackled, they

* Italics are mine.

have supplied elaborate explanation. The reader can test the validity of this statement from the works of Bhāskarācārya when he tackles the indeterminate equations. The only thing is that we are accustomed to view the old Mathematics in the light of the modern mathematical logic, as we are taught that way. If approached in the proper Hindu way—the way in which the old Hindus looked at the whole thing—we would find that there is nothing more logical than the reasoning they advanced. The advance of any science is to be studied in the light of the historical progress that is being gradually piled up; if we attach the same strict rules of Modern logical reasoning, Indian Mathematics—why even the Greek Mathematics—is bound to fall short and would not satisfy our requirements.

Again it would be observed that many European Scholars have tried to place some of the works of the mediæval period to a later date whenever no mention of the date of its composition is made in the work. Such endeavours at times enable them to establish the comparative advance in Mathematics made in the countries of the Mediterranean basin. In fact the Europeans have learnt their Algebra and mensuration from the Arabs who borrowed it from the Hindus. In this connection note down the following statement अङ्कानां वामतो गतिः। *aṅkānāṁ vāmato gatiḥ*. To denote any quantity proceed from left to right. The people whose script is written from right to left, would have had the same way to write any quantity. But throughout they have followed the Hindu method of the place value notation. This fact alone explains that the Arabs had to pick up their knowledge of numbers from Indians. Alberuni who had to visit India in the eleventh century stayed in India to study Mathematics and Astronomy and later on from the court of Babylon a number of

scholars came to India to learn the Science of Mathematics. Thus through Arabs the place value notation together with the knowledge of numbers was imported into Europe.

Irrespective of such a controversy however the fact remains that the brilliant achievements the Hindus showed in Algebra and Astronomy during the mediæval period are equal to none and they excel all those attained by the westerner during the same period.

For instance, in *Līlāvatī* which was written in 1136 by Bhāskarācārya we find a perfectly connected structure of science comprehending propositions which in Europe were invented successively by M. Bachet de Meziriac, Euler, and De La Grange from 17th century onwards; would it be justifiable to say that solutions given by this galaxy of Mathematicians were borrowed from Bhāskarācārya "through imperfect channels of communication" as Cajori puts it?* Just as such a statement would be misguiding and utterly foolish, similarly to conclude—that too, without any basis—that the Hindus borrowed the knowledge of mensuration from the Mediterranean 'through imperfect channels communications' would be at the face of it utterly misleading.

In this mediæval period, there lived a number of Mathematicians whose works were studied throughout

* Newton is credited for the invention of Calculus along with Leibnitz. Newton's work is said to be independent of that of Leibnitz. Both of them were contemporaries and belonged to the countries that are separated by a channel of hardly twentyseven miles. The communications between the two countries were at the time decidedly better than those existing between Greece and India. In the same tone and style as Cajori's would it be bold to remark that any one of them borrowed from the other through the *perfect* "channels of communication?

the length and breadth of India. Of all of these it is not possible to deal with all the work as such and therefore an attempt would be made to bring out the most important results achieved by them.

In order to gauge the progress in Mathematics in India during this mediæval period, it would be sufficient to treat only the momentus results achieved by the four Mathematicians, Āryabhaṭṭa I, Brahmagupta, Mahāvīr and Bhāskarācārya, both from the point of view of History of Mathematics and the developments in Mathematics. Though there lived a number of Mathematicians whose works have influenced those of some of the above mentioned celebrities, still it would be found that to the forward world march of the science their quota is but little. Of course their importance cannot be lost sight of, since they helped to keep up the continuity of the progress and thus in a way paved the path of these galaxy of Mathematicians enabling them to discover most wonderful results which have become the pride and glory that is Ind.

I cannot however permit myself to continue these reflections any further. The main purpose of this booklet is to piece together some features which indicate the developments made by the Ancient Indians. This having been done the lessons and the conclusions to be drawn must be allowed to remain over for discussion and consideration for impartial and intelligent readers.

CHAPTER II

DEVELOPMENT IN LOGISTICS IN VEDIC AGE

At the outset a few references from the Vedas would prove that the Hindus had an elaborate system of numerals.

> आ—विंशत्या त्रिंशत्या याह्यर्वाङ्चत्वारिंशता हरिभिर्युजानः ॥
>
> आ—पञ्चाशता सुरथेभिरिन्द्राह्या सप्तत्या सोमपेयम् ॥
>
> आ—शित्या नवत्या याह्यर्वाङ्शतेन हरिभिरुह्यमानः ।०॥

Ṛgveda II 18, 4 to 6.

> एका च दश च, दश च शतं च शतं च सहस्रं च सहस्रायुतं च अयुतं च नियुतं च, नियुतं च प्रयुतं च प्रयुतं च अर्बुदं च, अर्बुदं च न्यर्बुदं च समुद्रश्च मध्यं च अंतश्च परार्धश्च

Vājasaneyī Saṁhitā XVII 2.

There are a number of such quotations and so only the references would be sufficient.

In the seventh Kāṇḍa (part) in second Prapāṭhak (subpart) anuvāk (अश्वमेधांग रात्रिहोंम मन्त्रा संख्यादेवताकाः) 11 to 20 of Taittirīya Saṁhitā while offerings are made to the sacrificial fire all numerals are given and odd and even numbers upto hundred are repeated. They called an odd number Ayugma and an even number Yugma. In addition they contain the tables for 4, 5, 10, 20, 100 and they are given in the following order.

$4\times1=4$	$5\times1=5$	$10\times1=10$	$20\times1=20$
$4\times2=8$	$5\times2=10$	$10\times2=20$	$20\times2=40$
$4\times3=12$	$5\times3=15$	$10\times3=30$	$20\times3=60$
	...	...	$20\times4=80$

$4 \times 25 = 100, \cdot 5 \times 20 = 100, \; 10 \times 10 = 100, \; 20 \times 5 = 100$

$100 \times 1 = 100$	$100 \times 1 = 100$
$100 \times 2 = 200$	$100 \times 10 = 1000$
$100 \times 3 = 300$	$100 \times 100 = 10000$
... ...	... ...
$100 \times 10 = 1000$	$100 \times 10^{10} = 10^{12}$

Moreover in the 20th anuvāk (Taittirīya Saṁhitā VII Kāṇḍa) the following definitions are introduced

10^2	They	called	Śata	10^{11}	They	called	Anta
10^3	,,	,,	Sahasra	10^{12}	,,	,,	Parārdha
10^4	,,	,,	Ayuta	10^{13}	,,	,,	Uṣas
10^5	,,	,,	Niyuta	10^{14}	,,	,,	Vyuṣṭi
10^6	,,	,,	Prayuta	10^{15}	,,	,,	Deśyat
10^7	,,	,,	Arbuda	10^{16}	,,	,,	Udyat
10^8	,,	,,	Nyarbuda	10^{17}	,,	,,	Udit
10^9	,,	,,	Samudra	10^{18}	,,	,,	Suvarga
10^{10}	,,	,,	Madhya	10^{19}	,,	,,	Loka.

Such systematic definitions establish the fact that the Hindus had a definite system of denoting any number upto 10^{19}. Besides they proved that they were conversant at least with addition and multiplication—the fundamental operations of Arithmetic.

After the Vedas we come to Vedāṅgas in which Vedāṅgajyotiṣ forms an important part. Almost each and every verse of the Vedāṅgjayotiṣ convinces us that in addition to the knowledge of numerals, Hindus were well versed in logistics—addition, subtraction, multiplication, division, four types of examples in vulgar fractions and the rule of three. A few verses would suffice to prove the validity of the statement.

Consider the verse

निरेकं द्वादशाभ्यस्तं द्विगुणं गतसंयुतम् ।
षष्ठ्या षष्ठ्या युतं द्वाभ्यां पर्वणां राशिरुच्यते ॥

IV Ārcajyotiṣ.

Prose order (अन्वयः) निरेकं द्वादशाभ्यस्तं गत (सौरमास) संयुतं द्विगुणं (सौरपर्वाणिराशिः) । षष्ठ्या षष्ठ्यायुतं द्वाभ्यां (चांद्र) पर्वाणां राशिरुच्यते ।

Translation :—Subtract one from the number of solar years. Multiply it by twelve, add the number of past solar months and multiply the sum by two (and we get solar *parvas*). For every sixty of these sixty-two are the lunar parvas.

Explanation :—Let r be the complete solar months of the n^{th} year. Then the above rule can be summed up as

$$2\left[12\,(n-1)+r\right] \text{ are the solar parvas}$$

while the lunar parvas are given by

$$\frac{62}{60}\left[2\left\{12\,(n-1)+r\right\}\right].$$

To understand these formulæ, it is sufficient to know that the Hindus of the time had introduced the following definitions

2 parvas = 1 month
12 months = 1 year
and 5 years = 1 yuga or cycle.

Hence in one yuga there are 60 solar months while they used to take in one yuga 62 lunar months. Hence in order to find the number of parvas that elapsed at the end of r months of the n^{th} year we have first to subtract one from n so that it gives $(n-1)$ years that have elapsed.

Multiplying this by12 we get $12(n-1)$ months. Addition of r to this gives us $[12(n-1)+r]$ the total months that have elapsed. Double of this would obviously be the number of solar parvas that have elapsed till then.

Again to convert solar parvas into the lunar ones, we have simply to follow the rule of three.

60×2 solar parvas are equivalent to 62×2 lunar parvas then $2[12(n-1)+r]$ solar parvas are therefore equivalent to $\frac{62 \times 2}{60 \times 2}\left[2\left\{12(n-1)+r\right\}\right]$ lunar parvas

$$\text{i. e. } \frac{62}{60}\left[\quad\left\{12(n-1+r\right\}\right] \text{ lunar parvas.}$$

That is how the composers of the Vedāṅga jyotiṣ must have arrived at the above formulæ.

On the whole, the formulæ require a thorough mastery over logistics. And the proficiency of the Hindus in the fundamental operations of Arithmetic is revealed in the derivation of the above results.

Consider also the X verse.

भांशास्स्युरष्टकाः कार्याः पक्षद्वादशकोद्गताः ।
एकादशगुणश्चोनं शुक्लेऽर्धे चैंदवा यदि ॥

X Ārcajyotiṣ.

Prose order (अन्वयः) द्वादशकोद्गताः पक्षाः भांशाः अष्टकाः कार्याः स्युः उनः एकादश गुणस्य अस्ति । शुक्ले यदि ऐंदवा अर्धं च ॥

Translation :—Parvas which are twelve or multiples of twelve in number have their Bhāṁśas (parts of Bha—constellation) eight or multiples of eight. For the rest (which are not twelve or multiples of twelve) multiply the number by eleven to obtain the Bhāṁśas. In bright half the month to find the moon's position in the constellation add 62 (half of 124 lunar parvas in a yuga) to obtain the Bhāṁśas.

Explanation:—The number of constellation is twenty-seven. The sun passes through twenty-seven constellations i. e. one complete circle in one year. Hence in five years the sun travels through 27 × 5 constellations. One cycle of five years contains 124 parvas. Hence in one parva the sun goes through $\frac{27 \times 5}{124}$ constellations $\left(1+\frac{11}{124}\right)$ i. e. one complete constellation together with 11 parts of the next if each constellation is divided into 124 parts. These parts or aṁśas of the constellation are called Bhāṁśas.

Hence to obtain the Bhāṁśas for any particular parva the number of that parva will have to be multiplied by 11 and divided by 124. After removing the integral part, the numerator of the fraction gives the Bhāṁśas for that parva. Thus for the n^{th} parva the number of constellations travelled by the sun are

$$n\left[1+\frac{11}{124}\right]=n+\frac{11n}{124}.$$

Let $\frac{11n}{124} = \mathrm{I} + \frac{\mathrm{B}}{124}$ where I denotes the integral part in $\frac{11n}{124}$.

Then

$$n\left(1+\frac{11}{124}\right)=\left(n+\mathrm{I}\right)+\frac{\mathrm{B}}{124} \qquad \text{.............. (A)}$$

Then B gives us the Bhāṁśas for the n^{th} parva. The first line of the above verse states that if n happens to be 12 or a multiple of 12, B is 8 or a multiple of 8.

If $n = 2r$ (i.e. even) the sun and the moon are in conjunction and so they have the same Bhāṁśas.

If $n = (2r - 1)$ (i. e. odd) which is so for the bright parva, the sun and the moon are in opposition with a difference of $\frac{27}{2} = 13\frac{1}{2}$ constellations.

Hence for the position of the moon the number of constellations travelled by the moon is obtained by adding $13\frac{1}{2}$ to (A). Hence it is

$$\left[2r + \mathrm{I} + 13 + \tfrac{1}{2}\right] + \frac{\mathrm{B}}{124} \qquad \text{when} \qquad n = 2r$$

and $$\left[(2r - 1) + \mathrm{I} + 13 + \tfrac{1}{2}\right] + \frac{\mathrm{B}}{124} \quad \text{when} \quad n = 2r - 1$$

i. e. $$\left[2r + \mathrm{I} + 13\right] + \frac{\mathrm{B} + 62}{124}.$$

or $$\left[2r - 1 + \mathrm{I} + 13\right] + \frac{\mathrm{B} + 62}{124}.$$

Hence the moon's Bhāṁśas are B + 62 i. e. B + half of 124 while those of the sun are B.

Such an elaborate and methodic analysis must have been developed by the Ancient Indians to arrive at the formulæ given in the above verse.

Again consider the verses XVI, XVII and the XVIII together.

कलादश सविंशास्याद्द्विमुहूर्तस्तु नाडिके ।
द्विर्त्रिंशस्तत्कलानां तु षट्शतीत्र्यधिकं भवेत् ॥
नाडिके द्वे मुहूर्तस्तु पञ्चाशात्पलमाषक ।
माषकात्कुंभको द्रोणः कुटपैर्वर्धते त्रिभिः ॥
ससप्तैकं भयुक्सोमः सूर्योद्योर्नि त्रयोदश ।
नवमानिच पञ्चाह्नः काष्ठाः पञ्चाक्षराः स्मृताः ॥

Ārcajyotiṣ XVI to XVIII

Prose order (अन्वयः). (नाडिकायाः) कलाः दशसविंशास्यात् मुहूर्तस्य द्वे नाडिके, द्विर्त्रिंशः नाडिकाः दिनं, कलानां षट्शतीत्र्यधिकं दिनं करोति ।

द्वे नाडिके मुहूर्तः आषकं तु पञ्चाशतत्पलं (भवति) आषकात्कुंभकः तस्मात् द्रोणः त्रिभिः कुटपैः वर्धते ।

सोमः सप्तैकं भयुक् । सूर्याद्यूनित्रयोदश पञ्चाहः नवमानिच । काष्ठाः पञ्चाक्षराः स्मृताः ।

Translation :—A *nāḍikā* is equal to $10 + \frac{1}{20}$ *kalās*, two *nāḍikās* make one *muhūrta*; sixty *nāḍikās* make a day, so also 603 *kalās* make a day.

Two *nāḍikās* make a *muhūrta* fifty *palas* make one *āṣaka*, from *āṣaka* kumbhaka is derived. Droṇa is greater than *kumbhaka* by three *kuṭapa*.

The moon is in one constellation for one *sāvan* day and seven kalās. The sun is in one constellation for $13\frac{5}{9}$ *sāvan* days. *Kāsṭha* means the time required to pronounce five long syllables.

Explanation:—Here we observe that the first two verses together with the last part of the last verse give us only the definitions of the units of time–*kalā*, *nāḍikā*, *muhūrta*, *pala*, *āṣaka*, *kumbhaka* *Droṇa* and *kaṭap*—the last four have also been defined in an earlier verse.

One *saur* year consists of 366 days. Hence a cycle of five years contains $5 \times 366 = 1830$ sāvan days. In one cycle of five years, the moon travels 27×67 constellations this has been given from the verse 31 of the yājuṣa Jyotiṣ.

Hence to go through one constellation the number of days required is $\frac{366 \times 5}{67 \times 27} = \frac{610}{603} = 1 + \frac{7}{603}$.

i.e. one sāvan day and 7 *kalās* since 603 kalās make one day.

Similarly in a cycle of five years i. e. 1830 *sāvan* days the sun travels through 27×5 constellations. Therefore

to travel through one constellation, the sun takes $\frac{1830}{27 \times 5} = \frac{122}{9} = 13 + \frac{5}{9}$ *sāvan* days.

There are many such verses in the vedāṅgajyotiṣ in each of which we find a formula derived after immense calculations. A part from the astronomical value of these results, it must at least be said that the proficiency the ancient Indians showed in the field of Mathematics is worthy of the vedic civilization and the credit must go to them for being the originators of the science of Astronomy and Mathematics.

CHAPTER III

DEVELOPMENTS IN GEOMETRY IN VEDIC AGE

In the days of Vedic civilization, India was under the powerful sway of religion and the necessity of performing the sacrificial rites to please the gods, so that they should deserve the boons which the gods confer on, in return, led the Indians to choose the auspicious time for the sacrifice and to make the shape of the altars as accurate as possible so as to make their offerings acceptable to the gods. This fervour of religious belief made the Adhvaryus of the time to develop Astronony as we have seen in the previous chapter while the accuracy of the shape of the altars for the sacrificial fire made them discover theorems in Geometry. They have also obtained the value of $\sqrt{2}$ correct to five places of decimals, as it was essentially required in the building of sacrificial fires.

All these developments are recorded in the Śulva sūtras which are the supplements of the 'Kalpa' Vedāṅga. Among the Śulvasūtras that belong to Kalpasūtras the most important are those of Baudhāyana and Āpastamba. There is also a Śulvapariśiṣṭa belonging to the Śukla Yajurveda and is ascribed to Kātyāyana. The composers of the Śulvasūtras have first of all enumerated the rules for measuring, next they teach us how to fix up the right places for the sacred fires and then about the shapes of the different Vedis (altars) Saumiki Vedi, Paitṛki Vedi etc. Then follows the detailed description of the 'agni' the altar built of bricks. The shapes of various kinds of [illegible] have been given. They contain the following

figures—a rhombus, an isosceles trapezium, a square, a rectangle, a right angled triangle and an isosceles right angled triangle. Every altar was to be constructed to five layers of bricks so as to reach the height of a knee, every layer consisting of 200 bricks. They further furnish us with all the particulars of various 'citis' (altars) viz. Śyenacit, Alajacit, Kaṅkacit, Smaśānacit etc. For all these citis, the area fixed was seven and a half square *puruṣas*. At the second construction, the area of the citi is to be increased by one square puruṣa, at the third construction by two square puruṣa and so on till finally for the *soma* sacrifice it is increased by twelve square *puruṣas*. Hence the rules for adding two squares were obtained. Again when they had a discussion regarding the construction of Gārhapatya altar, some opined that it should be of a square shape while the others asserted that it must have a circular shape and consequently there arose the need for circling the square and vice versa since in both the cases the area is to remain unaltered. Their aim was primarily religious and not mathematical, yet the brilliance they have shown is an index of the Mathematical developments of the age.

They have dealt with the following theorems.

(i) The wellknown theorem on a right angled triangle—i.e. the Śulva theorem (now wrongly known as Pythagoras theorem) with all its accompanying properties.

(ii) Drawing a perpendicular to a given line at a given point of it and how to draw the perpendicular bisector of any line.

(iii) Finding the mean proportional i.e. finding the side of a square equal in area to a rectangle.

(iv) To construct an isosceles trapezium equal in area to a square.

(v) The value of $\sqrt{2}$ correct to five places of decimals.

(vi) Find the circle equal in area to a given square and its converse.

We shall deal with these cases one by one.

I

Śulva (Pythagoras) Theorem

Baudhāyana and Kātyāyana state

दीर्घचतुरस्रस्याक्ष्णयारज्जुः पार्श्वमानी तिर्यङ्मानी यत्पृथग्भूते कुरुतस्तदुभयं करोति ।

Explanation :—दीर्घचतुरस्रस्य क्षेत्रस्य या तिर्यङ्मानीसाऽन्यत्र करणीत्वेन पृथग्भूता सती यत्प्रमाणकं समचतुरस्रं करोति तथा दीर्घ चतुरस्रस्य क्षेत्रस्य या पार्श्वमानी साऽन्यत्र करणीत्वेन पृथग्भूता सती यत्प्रमाणकं समचतुरस्रं करोति तदुभयमपि मिलितं दीर्घचतुरस्रस्याक्ष्णयारज्जुः करोति इति क्षेत्रज्ञानम् ।

Translation :—The cord (of the length of the) diagonal of a rectangle makes both (areas), which cords forming the length and breadth make separately.

Āpastamb states the same rule as:—

दीर्घस्याक्ष्णयारज्जुः पार्श्वमानी तिर्यङ्मानीच यत्पृथग्भूते कुरुतस्तदुभयं करोति ।

Thus they state that the square on the diagonal of a rectangle is equal to the sum of the squares on the adjacent sides.

The particular case of this theorem is given.

Baudhāyana :—समचतुरस्रस्याक्ष्णयारज्जुर्द्विस्तावतींभूमिं करोति ।

Āpastamba :—चतुरस्रस्याक्ष्णयारज्जुर्द्विस्तावतीं भूमिं करोति ।

Kātyāyana :—समचतुरस्रस्याक्ष्णया रज्जुर्द्विकरणी ।

Explanation :—समचतुरस्रस्य क्षेत्रस्य याऽक्ष्णया—कर्णसूत्रभूता या रज्जुः सा द्विकरणी—द्विगुणक्षेत्रस्य प्राची सूत्रस्य प्रमाणमित्यर्थः ।

Translation :—The cord of the length of the diagonal of a square makes an area of double the size.

Then the Sūtrakāras proceed to give some illustrations of the theorem.

Baudhāyana :—त्रिचतुष्कयोर्द्वादशिकपञ्चकयोः पञ्चदशिकाष्टिकयोः सप्तिक चतुर्विंशकयोर्द्वादशिकपञ्चत्रिंशिकयोः पञ्चदशिकषट्त्रिंशिकयोरित्येतासूपलब्धिः।

i. e. it states that

$$3^2 + 4^2 = 5^2$$
$$12^2 + 5^2 = 13^2$$
$$15^2 + 8^2 = 17^2$$
$$7^2 + 24^2 = 25^2$$
$$12^2 + 35^2 = 37^2$$
$$15^2 + 36^2 = 39^2$$

The last is a particular case of

$$12^2 + 5^2 = 13^2$$

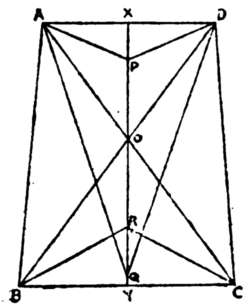
In addition, Āpastamba gives one more

$$15^2 + 20^2 = 25^2$$

which is also a particular case of

$$3^2 + 4^2 = 5^2.$$

All these cases of right angled triangles are required in *smaśāna cit* which has for its base an isosceles trapezium given below.



ABCD is an isoseceles trapezium with AD and BC as parallel sides. AD and BC are 24 and 30 units in length X, Y are the midpoints of AD and BC respectively and XY = 36 units. Since the trapezium is isosceles XY is perpendicular to AD and BC. AC and BD intersect at O. O would obviously be on XY such that

$$\frac{YO}{XO} = \frac{15}{12} = \frac{5}{4} \quad \therefore \quad \frac{YO}{XY} = \frac{5}{9}. \text{ But } XY = 36.$$

$$\therefore \quad YO = 20.$$

On XY the points P, Q, R are taken such that XQ = 35, XP = 5, and XR = 28. Then we get a set of right angled triangles.

(i)	$\triangle$APX or $\triangle$DPX	with sides	(5, 12, 13)	units
(ii)	$\triangle$AOX or $\triangle$DOX	with sides	(12, 16, 20)	units
(iii)	$\triangle$BRY or $\triangle$CRY	with sides	(8, 15, 17)	units
(iv)	$\triangle$BOY or $\triangle$COY	with sides	(15, 20, 25)	units
(v)	$\triangle$AQX or $\triangle$DQX	with sides	(12, 35, 37)	units
(vi)	$\triangle$BXY or $\triangle$CXY	with sides	(15, 36, 39)	units

Kātyāyana furnishes us with a few more examples

(a) पदं तिर्यङ्मानी त्रिपदा पार्श्वमानी तस्याक्ष्णया रज्जुर्दशकरणी ।

i. e. $1^2 + 3^2 = 10$

(b) द्विपदा तिर्यङ्मानी षट्पदा पार्श्वमानी तस्याक्ष्णयारज्जुश्चत्वारिंशत्करणी ।

i. e. $2^2 + 6^2 = 40$

Again द्विप्रमाणाचतुःकरणी त्रिप्रमाणा नवकरणी चतुःप्रमाणा षोडश करणी.

Translation :—A cord of double the length makes four squares, a cord of treble length makes nine (squares) and a cord four times the length makes sixteen (squares)

Again Kātyāyana and Āpastamba state

(a) अर्धप्रमाणेन पादप्रमाणं विधीयते.।

By a measure of half a length of the cord is made (square) equal to one-fourth (in area) i. e. $(\frac{1}{2})^2 = \frac{1}{4}$

Āpastamba :—

(b) तृतीयेन नवमीकला ।

or Kātyāyana :—तृतीयेन नवमोंऽशः ।

The third part (of the cord) makes (square equal to) one-ninth (in area) i. e. $(\frac{1}{3})^2 = \frac{1}{9}$

(c) Kātyāyana :—चतुर्थेन षोडशीकला

Translation:—The sixteenth part (of the area) is made by one-fourth (of the cord). i. e. $(\frac{1}{4})^2 = \frac{1}{16}$.

The statements (a), (b) and (c) establish the fact that they were fully conversant with the fractions and wherever fractions are used in śulvasūtras, it will be found that their noteworthy character is that they appear with unit numerator. Next there are given rules for combining the two equal and unequal squares.

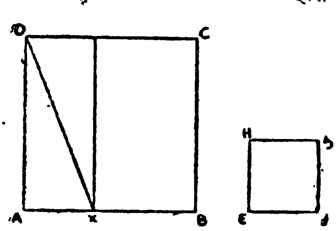
Āpastamba:—तुल्ययोश्चतुरस्रयोरुक्तः समासः । नानाप्रमाणयोश्चतुरस्रयोः समासः । ह्रसीयसः करण्या वर्षीयसो वृध्रमुल्लिखेत् । वृध्रस्याऽक्ष्णयारज्जुरुभे समस्याति ।

Baudhāyana:—नाना चतुरस्रे समस्यनकनीयसः करण्या वर्षीयसो वृध्रमुल्लिखेत् वृध्रस्याक्ष्णयारज्जुः समस्तया पार्श्वमानी भवति ।

Kātyāyana:—समचतुरस्राणामुक्तः समासः । नानाप्रमाण समासे ह्रसायसः करण्या वर्षीयसोऽपच्छिन्धात्तस्याक्ष्णयारज्जुरुभे स्यतीति समासः ।

Explanation:—नानाप्रमाणम्—भिन्नप्रमाणानाम् समचतुरस्राणां समासे संक्षेपे कार्ये ह्रसीयसो—ह्रस्वतरस्य समचतुरस्रस्य करण्यावर्षीयसो वृद्धतरस्य करणीं पार्श्वमानी तिर्यङ्मानी वाऽपच्छिन्द्यात्-ह्रस्व-करणीमानेन दीर्घकरणीं कोणतोमित्वां विदारायित्वा अविशिष्टं जह्यात् । तस्य ह्रस्वकरणीमानेन छिन्नस्य दीर्घक्षेत्रस्याक्ष्णया रज्जुः करणीभूय अन्य चतुरस्रं कुर्वतीउभे—वृद्धह्रस्वतरे क्षेत्रे समस्यति संक्षिपति ।

इति समासः ।



Let ABCD and EFGH be the two squares to be combined, Let AX be equal to EF. Then join DX and the square with the side DX is the square equal to the sum of the squares ABCD and EFGH.

For,

$$AX^2 + AD^2 = DX^2$$
$$EF^2 + AD^2 = DX^2.$$

II

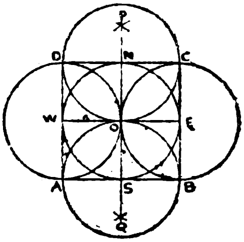
To find the perpendicular bisector

In order to draw a square i. e. the problem of drawing a perpendicular to a given line, various methods have been given by Āpastamba, Kātyāyana and Baudhāyana which are directly based upon the śulva theorem. They afford the illustrations of the sum of two squares equal to a third one, beginning with $3^2 + 4^2 = 5^2$. These illustrations of the sum of two squares have already been stated above. But there is one method which is very ingenious and at the same time seems to be the most ancient of all methods given in the sūtras and is found in Baudhāyana only. It is as follows :—

चतुरस्रं चिकीर्षन्यावच्चिकीर्षेत्तावती रज्जुमुभयतः पाशां कृत्वा मध्ये लक्षणं करोति। लेखामालिख्य तस्या मध्ये शङ्कू निहन्यात्तस्मिन्पाशौ प्रतिमुच्य लक्षणेन मण्डलं परिलिखेत्। विष्कंभान्तयोः शङ्कूनिहन्यात् । पूर्वस्मिन्पाशं प्रतिमुच्य पाशेन मण्डलं परिलिखेत्। एवमपरस्मिँस्ते यत्र समेयातां तेन द्वितीयं विष्कंभमायच्छेत्। विष्कंभात्तयोः शङ्कू निहन्यात्। पूर्वस्मिन्पाशौ प्रतिमुच्य लक्षणेन मण्डलं परिलिखेत्। एवं दक्षिणत एवं पश्चादेवमुत्तरतस्तेषां येन्त्याः संसर्गास्तच्चतुरस्रँ संपद्यते।

Translation:—If you wish to draw a square, take the cord of the desired length, tie at both ends and mark its middle (point) and fix a pole there, and draw the prācī line (lekhā). Having fastened (midpoint of the cord) at the pole the two ties (at the ends) describe with the mark a circle round it. Then fix the poles at both ends of the diameter (formed by the prācī line) and fastening one tie at the eastern pole (the pole standing at the east end of the prācī line) describe a circle with the other tie (i. e. with the full length of the cord). In the same manner a circle is described round the pole at the western end (of the prācī) and joining the points of intersection

of the circles, another diameter of the first circle is obtained along this line. (This is the diameter along the line due north-south). Fastening both the ties at the eastern pole describe a circle round it with the mark. The same (operation) is to be done in the south, the west and the north (i. e. the circles are to be described round the other three poles).[4] The points of intersection of these circles are the four corners of the required square.



Let $2l$ be the length of the side of the required square. Take any point O and with O as centre and l as radius describe a circle and take its diameter along the prācī line, i. e. WE is that diameter, E being the eastern point and W the western one. With E and W as centres and radius $2l$ describe circles that will cut each other in P and Q. PQ runs along the north south line and obviously bisects WE at O. NS is the diameter of the first circle running along PQ. With E, N, W, S, as centres and l as radius describe four circles. These intersect in points A, B, C and D. Then ABCD is the required square.

The ingenuity of this construction lies in first finding the line NS perpendicular to WE. Though ancient in its way the method is accurate in its construction and show the triumph of mathematical genius.

III

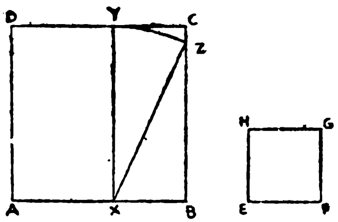
To find the mean proportional

To find the difference of two squares, and hence to derive the mean proportional.

Baudhāyana and *Āpastamba* :—चतुरास्रच्चतुरस्रं निर्जिहीर्षन्यावत्रिः जिहीर्षेत्तस्य करण्या वर्षीयसो वृध्रमुल्लिखेद्वृध्रस्य पार्श्वमानीमक्ष्णयेरतत्पार्श्वमुपसंहरेत्मा निपतेत्तदपच्छिन्द्याच्छिन्नया निरस्तम् ।
or in Kātyāyana's words चतुरस्राच्चतुरस्रं निर्जिहीर्षन्यावन्निर्जिहीर्षेत्तावदुभयोऽपच्छिन्द्य शङ्कू निरखायपार्श्वमानीं कृत्वा पार्श्वमानींसाम्मितामक्ष्णयां तत्रौपसंहरति समासेऽच्छेद सकरध्येष निर्ह्रासः ।

Explanation :—तत्र बृहत्प्रमाणक समचतुरस्रस्य एकपार्श्वगत कोणाभ्यां संमुख भुजयोरल्प चतुरस्रभुजतुल्ये रेखे अपिछिन्द्य तच्छेदेन बिन्दुद्वये शङ्कू निखाय शङ्कू मुलयोर्बृहत्सम चतुरस्रभुजतुल्य सूत्रं बध्वा बृहचतुरस्रभुजतुष्यदैर्घ्ये एककोणाद्बृहद्समचतुरस्रभुजतुल्यां रेखां तथा स्थापयेत् यथा तदग्रं सम्मुखपार्श्वमान्यां लगति ।

Translation :—If you wish to deduct one square from the other cut off from the larger one a rectangle with the side of the smaller one, draw one side of that rectangle across to the other side where it touches the other side that piece is cut off, by it the deduction is made.



Let ABCD and EFGH be two squares. Cut off $XB = EF$.

Draw $XY \perp$ to AB cutting CD in Y. Rotate XY about X till it cuts BC in Z such that $XZ = XY$

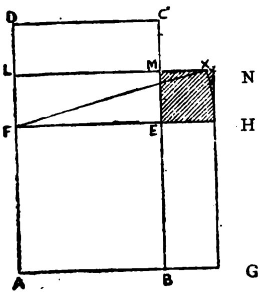
Then $XZ^2 - XB^2 = BZ^2$

This proposition is made use of to obtain the mean proportional.

Baudhāyana :—दीर्घं चतुरस्रं चिकीर्षंस्तिर्यङ्मानी करणीं कृत्वा शेषं द्वेधा विभज्य विपर्यस्येतरमोपदध्यात् खण्डभावापेनतत्सपूरयेतस्य निर्हार उक्तः।
or in Kātyāyana's words दीर्घचतुरस्रं समचतुरस्रं चिकीर्षन्मध्ये तिर्यगपच्छिद्यान्यतर द्विभज्येतरत्पुरस्ताद्दक्षिणतश्चोपदध्याच्छेषमागन्तुना पूरयेत्तस्योक्तः निर्ह्रासः।

Explanation :—अत्र आयामार्द्धविस्तारं चतुरस्रं दीर्घं चतुरस्रपदेन व्यवह्रियते। दीर्घं चतुरस्रंविस्तार द्विगुणायायं चतुरस्रसमचतुरस्रं कर्तुंविच्छन् मध्ये तिर्यग्रज्जुं प्रसार्यापच्छिन्द्य-विभज्य आयामस्य तुष्यभागद्वयं कृत्वा अन्यतराद्विभज्य— तत्र एकतरस्य समचतुरस्रस्य पुनस्तथैव भागद्वयकरणेन द्वे दीर्घं चतुरस्रे विधाय इतरत्—भागद्वयं पुरस्ताद्दक्षिणतश्चोपदध्यात् एकं भागं दीर्घचतुरस्ररूपं पूर्वत्र आविभक्त समचतुरस्रे तिर्यगायाभं कृत्वा योजयेत् शेषम्—समचतुरस्रे क्रियमाणे न्यूनं नन्वेवं समचतुरस्रमधिकं जातं तस्योक्ता निर्ह्रासः "चतुरस्राच्चतुरस्रं निर्जिहीर्षन्" इति सूत्रे।

Translation :—In order to turn a rectangle into a square take the breadth for the side of the square, divide the rest of the rectangle into two equal parts and inverting their places join these two parts to two sides of the square. Fill the empty space with the added piece: (thus the problem is reduced to the problems of finding the difference of two squares and) the deduction of this is given.



ABCD is a rectangle, AB being the breadth and AD the length. Complete the square ABEF. Bisect the remaining rectangle FECD by LM and keep the part LMCD by the side BE so that its new position is BGHE. Complete the square (fill the square) EHNM. Then from the square AGNL subtract the square EHNM. The method of obtaining the difference of two squares is already given. It is as follows: with F as centre and radius FH cut off

LN at X (see the above proposition). Then LX is the required side of the square equal in area to the rectangle.

This method of obtaining the mean proportional is similar to the modern method. Here LX is the mean proportional of AD and AB for

$$LX^2 = FX^2 - LF^2 = AG^2 - BG^2$$
$$= (AG + BG)(AG - BG)$$
$$= AD.\ AB$$
$$\text{If } a = AD,\ b = AB$$
$$AL = b + \frac{a-b}{2} = \frac{a+b}{2}$$
$$DL = BG = EH = \frac{a-b}{2}$$
$$\therefore \left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2 = ab.$$

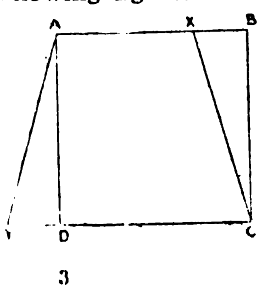
IV

To construct an isosceles trapezium equal in area to a square

Baudhāyana —चतुरमेकतोऽणिमाश्चिकीर्षन्नणिमतः करणीं तिर्यङ्मानी कृत्वा शेषमक्ष्णया विभज्य विपर्यस्येतरत्रोऽपदध्यात् ।

Translation :—If you wish to make a trapezium from a square, mark the required length (on one side) and draw the diagonal and reversing (the triangular piece thus cut) join it to the other (side).

The meaning of it would be quite clear from the following figure.



Let ABCD be a square. Mark AX equal to the top of the required trapezium. Join CX. Remove the triangle CXB and place it on AB such that C falls on A and CB along AD while X takes the position Y off C and D.

V

The value of $\sqrt{2}$

Baudhāyana and Āpastamba :—प्रमाणं तृतीयेन वर्धयेत चतुर्थे-नात्मचतुस्त्रिंशोनेन सविशेषः ।

Kātyāyana :—करणी तृतीयेन वर्धयेतच्च स्वचतुर्थेनात्मचतुस्त्रिंशोनेन सवि-शेष इति विशेषः ।

Explanation :—करणीं—क्षेत्रायामं तृतीयेन—स्वतृतीयांशेन वर्धयेत् । तदिति सामान्ये नपुंसकम् । तत्—तं तृतीयांशं स्वचतुर्थेन—स्वस्य तृती-यांशस्य यश्चतुर्थोभागः तेन वर्धयेत् । किं भूतेन स्वचतुर्थेन, आत्मचतुस्त्रिं-शोनेन—आत्मनः चतुर्थांशस्य यश्चतुस्त्रिंशोभागः तेन ऊनेन हीनेन रहि-तेन, अर्थात् स्वचतुस्त्रिंशांशहनिचतुर्थभागेन तृतीयांशं वर्धयेत् ।—भेदकः करणी द्विकरण्योरिति विशेषः ।

Translation :—Increase the measure by its third part and this third by its own fourth part less the thirty fourth part of that fourth. This is saviśeṣa (सविशेष). If *karaṇī* the side of the square be one, *dvikaraṇī* the measure of the diagonal is given by

$$\sqrt{2} = 1 + \frac{1}{3} + \frac{1}{3 \cdot 4} - \frac{1}{3 \cdot 4 \cdot 34}.$$

This value of $\sqrt{2}$, when worked out would be found to be correct to five places of decimals. The question is how is it arrived at? The explanation given by Prof. Thibaut * as to the probable calculations the sūtrakāras might have done, fails to satisfy us.

He thinks that they must have taken squares of sides 1, 2, 3......etc. upto 20 units and written down the squares of the diagonals and compared them with the nearest square number. Thibaut thus gives the following table :

* Asiatic Journal of Bengal vol. XLIV 1875.

Side	Square of the diagonal	Nearest square number.
1	2	$1^2=1$
2	8	$3^2=9$
3	18	$4^2=16$
4	32	$6^2=36$
5	50	$7^2=49$
6	72	$8^2=64$
7	98	$10^2=100$
8	128	$11^2=121$
9	162	$13^2=169$
10	200	$14^2=196$
11	242	$16^2=256$
12	288	$17^2=289$
13	338	$18^2=324$
14	392	$20^2=400$
15	450	$21^2=441$
16	512	$23^2=529$
17	578	$24^2=576$
18	648	$25^2=625$
19	722	$27^2=729$
20	800	$28^2=784$

"How far" states Prof. Thibaut, "they went in their experiments we are unable to say; the list upto twenty suffices for our purpose. The three cases occur in which the number expressing the square of the diagonal of a square differs only by one from a square number. The cases are 8–9, 50–49, 288–289 the last case being the most favourable as it involves the largest number. The diagonal of a square the side of which is equal to 12 is very little shorter than 17 ($\sqrt{289}=17$). Would it then not be possible to reduce 17 in such a way as to render the square of the reduced number equal or almost equal to 288? Suppose they drew a square the side of which was 17 *padas* long and divided it into $17 \times 17=289$ small squares.

If the sides of the squares could now be shortened by so much, so that its area would contain not 289 but 288 such small squares, then the measure of the side would be the exact measure of the diagonal of a square, the side of which is equal to 12 ($12^2+12^2=288$). When the side of the square is shortened a little, the consequence is that from the sides of the square, a strip is cut off, therefore a piece of that length had to be cut off that the area of the two strips would be equal to one of the 289 small squares. Now as the square is composed of 17×17 squares one of the two strips cuts off a part of 17 small squares and the likewise of 17, both together 34 and since these 34 cut off pieces are to be equal to one of the squares, the length of the piece to be cut off from the side is fixed thereby, it must be 34th part of the side of one of the 289 small squares.

The thirtyfourth part of the 34 small squares being cut off and the area of the large square reduced to 288 small squares, the two strips which are cut off from the sides of the square, let us say the east side and the south side, intersect or overlap each other into the smallest corner and the consequence is that from the small square in that corner, not $\frac{2}{34}$ are cut off but only $\frac{2}{34}-\frac{1}{34\times 34}$ hence the error in the determination of the value of Ṣaviśeṣa."

In justification of his ingenious interpretation, Prof. G. Thibaut quotes an example given by commentators of the sūtras viz. the square whose side is 12 and the diagonal is $16\frac{33}{34}$. For, $16 = \left(12+\frac{12}{3}+\frac{12}{3\cdot 4}-\frac{12}{3\cdot 4\cdot 34}\right)$ approximately.

But this example given by commentators of later date is indeed the case recommended itself by being first in

which the third part of the number, and the fourth part of the third part are both whole integral numbers.

However, this clever and ingenious explanation by Thibaut fails to give the reason of the presence of the fraction $\frac{1}{3}$ and $\frac{1}{12}$ which occur as the second and the third term. It can no doubt make a suggestion (though not plausible) for the presence of the number 34.

G. R. Kaye states (Indian Mathematics by Kaye) that "given scale of measure based upon the change ratio 3, 4 and 34 (since they had a scale 34 *tilas*=1 *aṅguli*) the result is only an expression of direct measurement." In my opinion Mr. Kaye has put the cart before the horse. Even accepting such an erroneous statement, the question still remains, why did they choose such a scale or what led them to have such a scale of units? Prof. Thibaut observes: "Nothing in the Sūtras would justify the assumption that they were expert in long culculations," and Kaye conveniently quotes only these words of Thibaut to qualify his statement and purposely avoids a further statement by Thibaut which is not only an answer to Kaye, but Thibaut thereby refutes his own statement. For he further says, "As the diagonal of a square is in reality an incommensurable quantity we can expect an approximate value but their approximation is **remarkably closed one.*" No doubt in Sūtras they did not try to exhibit their expertness in long calculations, but it must be noted that Śulvasūtras are not mathematical works but works that had their origin in religious duties performed and, as such, there was no need of parading the above mentioned quality. They gave the value of $\sqrt{2}$ correct to five places of decimals, and in those days it was really an uncommon achievement. How did the scale 34 *tilas*=1 *aṅguli* and 12 *aṅgulis*=1 *prādeśa* came

* Italics are mine.

into existence? Out of all the numbers, why the number 34 was suggested to Baudhāyana—the author of the oldest Śulvasūtra—to define the measure *aṅguli* instead of any round figure? It is in fact, Baudhāyana came, in the calculations of $\sqrt{2}$, across the 34th part of $\frac{1}{12}$ that led him to introduce this scale of relation. Baudhāyana adopted the system of units befitting the scientific mind that harnesses theory with practice. For instance, why is the metric system of units introduced? Because it facilitates calculations in scientific experiments. The same genius that introduced the metric system was present in Baudhāyana when he used the value of $\sqrt{2}$ in defining his units and their measures. This is therefore the first attempt on the part of Baudhāyana to make the theory suit the practice and full credit must go to him for its priority.

It appears to my mind that this value of $\sqrt{2}$ must have been arrived at by the method of approximations where the quantities of higher order of smallness are neglected or by the method which wrongly goes under the name Tannery's R-process.

In this connection the reader must first of all note down an earlier Sūtra.

अध्यर्धपुरुषा रज्जुर्द्वौ सपादौ करोति

Translation:—A cord of the length of one and a half *puruṣa* makes two and a quarter square *puruṣas*.

This shows that the Sūtrakāras knew that the area of the square whose side is $(1+\frac{1}{2})$ units is $2\frac{1}{4}$ square units, *i. e.*,

$$(1+\tfrac{1}{2})^2 = 2\tfrac{1}{4} > 2.$$

Hence they tried the square of $(1+\frac{1}{3})$

$$\text{But} \quad (1+\tfrac{1}{3})^2 = \frac{16}{9} < 2$$

$$2 - \frac{16}{9} = \frac{2}{9}.$$

So they further took the quantity x, so that

$$(1+\tfrac{1}{3}+x)^2 = \frac{16}{9} + \frac{8x}{3} + x^2.$$

x^2 they omitted as the quantity of higher order of smallness. Hence putting

$$\frac{8x}{3} = \frac{2}{9},$$

we have $x = \dfrac{1}{12} = \dfrac{1}{3.4}$

$$\therefore \sqrt{2}=1+\frac{1}{3}+\frac{1}{3.4} \text{ as the second approximation.}$$

Further $\left(1+\dfrac{1}{3}+\dfrac{1}{3.4}\right)^2 = \left(\dfrac{17}{12}\right)^2 = \dfrac{289}{144} > 2$

$$2-\frac{289}{144} = -\frac{1}{144}.$$

So they further took the quantity y (say) such that

$$\left(1+\frac{1}{3}+\frac{1}{3.4}+y\right)^2 = \left(\frac{17}{12}+y\right)^2 = \frac{289}{144}+\frac{34y}{12}+y^2.$$

y^2 they omitted, and hence putting

$$\frac{34y}{12} = -\frac{1}{144}, \text{ we have}$$

$$y = \frac{-1}{3\cdot4\cdot34}$$

$$\therefore \sqrt{2}=1+\frac{1}{3}+\frac{1}{3\cdot4}-\frac{1}{3\cdot4\cdot34}$$

Or they might have followed another method that consists of splitting the quantity under the radical sign into two parts—one complete square and the other the remaining part, *i. e.*, the method that gives

$$\sqrt{A^2+b} = A + \frac{b}{2A} \text{ approximately.}$$

Thus

$\sqrt{2} = \sqrt{1+1} = 1+\tfrac{1}{2}$, as the first approximation.

But they knew that

$$\left(1+\frac{1}{2}\right)^2=\left(\frac{3}{2}\right)^2=\frac{9}{4}>2.$$

$\therefore$ They tried $\left(1+\frac{1}{3}\right)^2=\left(\frac{4}{3}\right)^2=\frac{16}{9}.$

But $\frac{16}{9}<2$ by $\frac{2}{9}$.

$$\therefore\ \sqrt{2}=\sqrt{\frac{16}{9}+\frac{2}{9}}=\sqrt{\left(\frac{4}{3}\right)^2+\frac{2}{9}}$$

$$=\frac{4}{3}+\frac{\left(\frac{2}{9}\right)}{2\cdot\left(\frac{4}{3}\right)}=\frac{4}{3}+\frac{1}{3\cdot4}$$

$=1+\frac{1}{3}+\frac{1}{3\cdot4}$, as the second approximation,

$$=\frac{17}{12}$$

And $\left(\frac{17}{12}\right)^2=\frac{289}{144}$

But $\frac{289}{144}<2$ by $\frac{1}{144}$

$$\therefore \sqrt{2}=\sqrt{\frac{289}{144}-\frac{1}{144}}=\sqrt{\left(\frac{17}{12}\right)^2-\frac{1}{144}}$$

$$=\frac{17}{12}-\frac{\left(\frac{1}{144}\right)}{2\times\left(\frac{17}{12}\right)}=\frac{17}{12}-\frac{1}{3.4.34}$$

$=1+\frac{1}{3}+\frac{1}{3.4}-\frac{1}{3.4.34}$ as the third approximation and so on.

Had they required the further approximations, they would have proceeded in the same manner but they did not find it necessary, as this value of "Saviśeṣa," *i. e.*, $\sqrt{2}$, served their purpose quite all right.

The second method is but a little variation of the first but it must have been this very method by which the Sūtrakāras obtained the value of $\sqrt{2}$ as it is in conformity with the word Saviśeṣa (सविशेष) :

सविशेष = स + वि + शेष

शेष—means a remainder or any thing left out to be said ;

वि—as a prefix to nouns not immediately connected with roots expresses negative sense and hence विशेष means without anything being left out to be said, *i. e.*, complete in itself—*i. e.*, a complete square ;

स—the prefix attached to nouns means 'the same'—'the celebrated'—'the distinguished.'

Hence सविशेष means the distinguished or the celebrated complete result.

विशेषितः शेषः तेन युक्तं—सविशेषितशेषः or सविशेषशेषः or for brevity सविशेषः

Thus in $\sqrt{2}$,

1, $\frac{16}{9}$, $\frac{289}{144}$ are the विशेष or perfect squares with 1, $\frac{2}{9}$, $-\frac{1}{144}$ as the corresponding शेष or remainders.

For example in $\sqrt{41}$ *i. e.* $\sqrt{36+5}$

36 is विशेष and 5 is शेष.

Thus it is evident that the process

$$\left[\sqrt{A^2+b} \sim \left(A + \frac{b}{2A}\right)\right]$$

must be known, because of its priority, as the Saviśeṣa process and not as Tannery's R-process. Saviśeṣa process is at least as old as 750 B. C., and bears a strong testimony

that the Hindus had made enormous strides in the science of Mathematics in ancient days.

VI

Finding the circle equal in area to a given square and the converse of it.

Baudhāyana :—चतुरस्रं मण्डलं चिकीर्षन्नक्ष्णयार्धं मध्यात्प्राचीमभ्यापातयेदतिशिष्यते तस्य सह तृतीयेन मण्डलं परिलिखेत् ।

Āpastamba gives the same rule in different words viz.

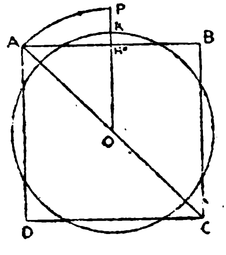
चतुरस्रं मण्डलं चिकीर्षन्मध्यात्कोट्यां निपातयेत् पार्श्वतः परिकृषातिशयेन सह मण्डलं परिलिखेत् । सानित्यामण्डलम् । (यावद्धियते तावदागन्तु)

or *Kātyāyana* चतुरस्रं मण्डलं चिकीर्षन्मध्यादंसे निपात्यपार्श्वतः परिलिख्य तत्र यदतिरिक्तं भवति तस्य तृतीयेन सह मण्डलं परिलिखेत् ।

Explanation :—चतुरस्रं क्षेत्रं मण्डलं कर्तुमिच्छन् मध्यात्—चतुर्भुज-मध्यात् अंसे—चतुर्भुजकोणे सूत्रं निपात्य दत्वा—चतुरस्रे सूत्रद्वये दत्ते सूत्रयोः संयोगस्थलं मध्यं स्यात् । ततो मध्याक्षरभ्य अंसान्तं पातिते अक्ष्णयार्धरूपे सूत्रे पार्श्वतः—पार्श्वमानेन करण्यर्धेन अङ्कयित्वा यदतिरिक्तम्—अङ्कनादधिको भागो भवति तस्य शिष्टस्य तृतीयैनांशेन सह करण्यर्धं संवर्ध्य तन्मानेन मण्डलं परिलिखेत् ।

Translation :—If you wish to turn a square into a circle stretch a cord of the centre (of the square) towards one of the corners, draw it round of a side and describe a circle together with the third part of the piece standing over.

To express it properly :—



Let ABCD be a square with centre O. Turn OA till it assumes the position OP ⊥ to AB, cutting AB in H. Take HK in HP equal to $\frac{1}{3}$ HP. Then with OK as radius, describe a circle which will be equal in area to the square.

Let us examine the value of π given by this construction. Let a be the side of the square.

Then OA $= \frac{1}{2}$ the diagonal of the square

$$= \frac{a\sqrt{2}}{2} = \frac{a}{\sqrt{2}}$$

$$\therefore \quad OK = \frac{a}{2} + \frac{1}{3}\left(\frac{a}{\sqrt{2}} - \frac{a}{2}\right)$$

$$\therefore 2\,OK = \frac{a}{3}\left(2 + \sqrt{2}\right)$$

$\therefore \quad d = \frac{a}{3}(2+\sqrt{2})$, where d is the diameter of the circle.

$$\frac{a}{d} = \frac{3}{2+\sqrt{2}}$$

$$\therefore \quad \frac{a}{d} = \frac{3(2-\sqrt{2})}{2} \qquad \dots \qquad \dots \qquad \dots \qquad (I)$$

Now $\frac{\pi d^2}{4} = a^2$ gives

$$\frac{a}{d} = \frac{\sqrt{\pi}}{2}$$

$$\therefore \quad \frac{\sqrt{\pi}}{2} = \frac{3}{2}(2-\sqrt{2})$$

$$\therefore \quad \sqrt{\pi} = 3\left[2 - \left(1 + \frac{1}{3} + \frac{1}{3.4} + \frac{1}{3.4.34}\right)\right]$$

$$\left(\because \quad \sqrt{2} = 1 + \frac{1}{3} + \frac{1}{3.4} - \frac{1}{3.4.34}\right)$$

$$\therefore \quad \sqrt{\pi} = \frac{3 \times 239}{12 \times 34}$$

$$\therefore \quad \pi = \frac{514089}{166464}$$

$$= 3{\cdot}088$$

Curiously enough, when the converse problem has been stated, the value of π, when worked out, turns out to be exactly the same. For the converse has been stated as :—

मण्डलं चतुरस्रं चिकीर्षन्विष्कंभमष्टौ भागान्कृत्वा भागमेकोनत्रिंशधा विभज्या-
ष्टविंशति भागानुद्धरेद्भागस्य च षष्ठमष्टभागोनम् ।

Translation:—If you wish to turn a circle into a square, divide the diameter into eight equal parts, and again one of those parts into twenty-nine parts; of these twenty-nine parts and the sixth part of the one left part less, with the eighth part of the sixth part.

Thus if a be the side of the square and d be the diameter of the circle,

$$a = d\left[1 - \frac{1}{8} + \frac{1}{8.29} - \frac{1}{8.29.6} + \frac{1}{8.29.6.8}\right] \quad \ldots \text{(II)}$$

Here again—

$$\frac{\pi d^2}{4} = a^2 \text{ gives}$$

$$\frac{\sqrt{\pi}}{2} = \frac{a}{d} = \left(1 - \frac{1}{8} + \frac{1}{8.29} - \frac{1}{8.29.6} + \frac{1}{8.29.6.8}\right)$$

$$\therefore \quad \sqrt{\pi} = \frac{9785}{5568}$$

$$\therefore \quad \pi = \frac{95746225}{31002624}$$

$$= 3{\cdot}088$$

There is much speculation as to the method Baudhāyana might have employed in establishing the relation (II) between a and d. Prof. Thibaut tries to give an explanation that is based upon the work of a commentator of a later date (Asiatic Journal of Bengal, 1875). His explanation is as follows :—

" Baudhāyana assumed $a=24$ *aṅgulis*, $\therefore \frac{a}{2} = 408$ *tilas* (as 34 *tilas*=1 *aṅguli*). Hence half the diagonal of the square

$$\frac{a}{\sqrt{2}} = 12\sqrt{2} = 12\left(1 + \frac{1}{3} + \frac{1}{3.4} - \frac{1}{3.4.34}\right)$$

$$= 16\frac{33}{34} = 16 \text{ aṅgulis and } 33 \text{ tilas}$$

$$\therefore \quad \frac{a}{\sqrt{2}} - \frac{a}{2} = 4 \text{ aṅgulis and } 33 \text{ tilas} = 169 \text{ tilas}$$

$$\therefore \quad \frac{1}{3}\left(\frac{a}{\sqrt{2}} - \frac{a}{2}\right) = 56\tfrac{1}{3} \text{ tilas}$$

$$\text{The radius of the circle} = a + \frac{1}{3}\left(\frac{a}{\sqrt{2}} - \frac{a}{2}\right)$$

$$= 408 + 56\tfrac{1}{3}$$

$$= 464\tfrac{1}{3} \text{ tilas}$$

" In other words if half the side of the square is 408 *tilas* long, the length of the radius of the circle amounts to 464⅓ *tilas*. In order to avoid the fraction, both the numbers were turned into thirds, the radius made=1393, and half the side=1224. Finally, the diameter was taken instead of the radius and the whole side of the square instead of half the side

" To generalise this rule, it was requisite to express 1224 in terms of 1393. One eighth of 1393 is equal to 174⅛, this multiplied by 7 is equal to 1218⅞. Difference between 1218⅞ and 1224 is 5⅛. Dividing 174 (Bāudhāyana takes 174 instead of 174⅛, neglecting the fraction as either insignificant or more likely as incovenient) by 29 we get 6. Subtracting from 6 its sixth part, we get 5 and adding to this the eighth part of this sixth part, we get 5⅛.

In other words

$$1224 = 1393\left(\frac{7}{8} + \frac{1}{8.29} - \frac{1}{8.29.6} + \frac{1}{8.29.6.8}\right)$$

Due allowance is being made for the neglected $\frac{1}{8}$".

To my mind it appears that this is not the method by which Baudhāyana might have arrived at such a wonderfully accurate result in consideration of the same value of π in both the cases.

The logical interpretation must have been as follows:

There is no need to take a square whose side is 24 *aṅgulis*.

For,

$$d = \frac{a}{3}(2 + \sqrt{2})$$

$$\therefore \frac{d}{a} = \frac{1}{3}\left[2+1+\frac{1}{3}+\frac{1}{3 \cdot 4} - \frac{1}{3 \cdot 4 \cdot 34}\right] = \frac{1393}{1224}$$

$$\therefore \frac{a}{d} = \frac{1224}{1393}.$$

To generalise this rule it was essential to express 1224 in terms of 1393. So while working it out, Baudhāyana found the greatest possible integral number that would divide both the numerator and the denominator giving small remainders. To obtain this number, divide 1393 by 1224, and the remainder is 169, the possible common measure for the two numbers 1393 and 1224. 1393 is divided by 169, the quotient is 8 and the remainder is 41: while if 1224 is divided by 169, the quotient is 7 and the remainder is 41. In the first case, to reduce the remainder to its lowest value by keeping quotient 8, and in the second case to reduce the remainder to its lowest value by keeping the quotient 7, the greatest common measure would be 174, since $174 \times 8 = 1392$, and

thus the remainder left is 1, and 174 × 7 = 1218, and leaves the remainder 6. Thus after 169, Baudhāyana must have by turn divided both the numbers by 170, 171, 172, 173 and 174, and stopped at 174, since it served the required purpose. This process involves simple division and therefore it is easily suggestible. Thus Baudhāyana obtained the maximum divisor for minimum remainders, since the idea was simply to express 1224 in terms of 1393. Thus—

$$\frac{a}{d}=\frac{1224}{1393}=\frac{\left[\frac{1224}{174}\right]}{\left[\frac{1393}{174}\right]}=\frac{\left[7+\frac{6}{174}\right]}{\left[8+\frac{1}{174}\right]}=\frac{7+\frac{1}{29}}{8+\frac{1}{6\cdot 29}}.$$

$$=\frac{\left[\frac{7}{8}+\frac{1}{8\cdot 29}\right]}{1+\frac{1}{6\cdot 8\cdot 29}}=\left[\frac{7}{8}+\frac{1}{8\cdot 29}\right]\cdot\frac{1}{\left[1+\frac{1}{6\cdot 8\cdot 29}\right]}$$

$$=\left[1-\tfrac{1}{8}+\frac{1}{8\cdot 29}\right]\times\frac{1}{1+\frac{1}{6\cdot 8\cdot 29}}$$

Now the value of the factor $\left[\frac{1}{\left(1+\frac{1}{6\cdot 8\cdot 29}\right)}\right]$ is obtained by simple division as in

$$\frac{1}{1+x}=1-x+x^2-x^3+\ldots\ldots\ldots\ldots$$

$$\therefore\ \frac{1}{1+\frac{1}{6\cdot 8\cdot 29}}=1-\frac{1}{6\cdot 8\cdot 29}+\left[\frac{1}{6\cdot 8\cdot 29}\right]^2-\left[\frac{1}{6\cdot 8\cdot 29}\right]^3+\ldots\ldots$$

$$\therefore\ \frac{a}{d}=\left[1-\tfrac{1}{8}+\frac{1}{8\cdot 29}\right]\left[1-\frac{1}{6\cdot 8\cdot 29}+\ldots-\right]$$

higher powers in the second bracket being neglected as they are small.

$$\therefore \frac{a}{d}=1-\frac{1}{8}+\frac{1}{8\cdot 29}-\frac{1}{6\cdot 29\cdot 8}+\frac{1}{8\cdot 29\cdot 6\cdot 8}-\frac{1}{8\cdot 29\cdot 6\cdot 29\cdot 8}$$

Omitting the last term as it is small, we have

$$\frac{a}{d}=1-\frac{1}{8}+\frac{1}{8\cdot 29}-\frac{1}{6\cdot 29\cdot 8}+\frac{1}{8\cdot 29\cdot 6\cdot 8}.$$

So it would be observed that by making use of simple division, the Sūtrakāras made use of the Binomial theorem with the negative unit index, and they were perfectly justified in doing so as the argument $x \left(=\frac{1}{6\cdot 8\cdot 29}\right)$ is less than unity. Throughout, this process is based upon ordinary division and as such it is easily suggestible.

CHAPTER IV

BAKṢALI MANUSCRIPT

After the Vedic age, we come to study the mathematics of the premediæval period. The greatest available work in the premediæval period is the Bakṣāli Manuscript. It is quite evident that much of the work in Mathematics that kept the continuity between the Vedic age the premediæval and mediæval periods has been lost and hence it goes very difficult to gauge the continuity of the progress done. Moreover India being a vast country almost a continent, the developments achieved in one part of the country reached late to the other part or at times did not reach at all. So the Mathematicians at times had to proceed to the study from the fundamentals of arithmetic to the advanced theory and were thus handicapped from the start. The means of communications and the absence of writing materials then were the greatest impediments in the path of the progress of every science that developed in Vedic age. It would be observed that even the work of the premediæval period was again lost and the Indian Mathematicians of the later age had to start again anew. That is exactly what has happened from the time of Āryabhaṭṭa onwards. The type of continuity we observe from the times of Aryabhaṭṭa I to the days of Bhāskarācārya and Nārāyaṇa cannot, in the absence of available manuscripts, be found in the developments from the Vedic age to the days in which the original Bakṣāli Manuscript was composed. But one thing is certain that the original Bakṣāli Manuscript shows as it were the capstone of the advance of the science of

Mathematics from the vedic age upto that period. Then came a lull. The old developments were lost or forgotten, or for want of sufficient number of persons who memorised it, the whole development was buried as it were, and a fresh attack on the field of Mathematics began. Of course the fundamentals of the science were there but the so far highly advanced theory was lost sight of and Āryabhaṭṭa, Brahmagupta, Bhāskara, Śrīdhara, Mahāvīr, and others have rediscovered the more advanced results obtained in Bakṣāli. In the Mediæval age there were communications, and however imperfect they may be, they helped to keep up the continuity. Unfortunately the Bakṣāli Manuscript does not seem to be known to Āryabhaṭṭa I and the later Mathematicians of India. Whatever knowledge they could get through imperfect channels of learning they did make use of. And that is why one or two of the problems—only the problems—of later Mathematicians bear a similarity to those in Bakṣāli Manuscript. They might have been even suggested to them even independently. On the score of this data the foreign scholars like Kaye make misleading statements and place the composition of the Bakṣāli Manuscript as late as the twelfth century. The circumstancial evidence together with the internal evidence makes us reject the conclusions drawn by Kaye as would be clearly seen in the following pages. Before proceeding with the discussion of the date it is desirable to know the history of the Manuscript copy of Bakṣāli.

The Bombay Govt. Gazette of 13th August 1881 gives the following account.

"The remains of a very ancient papyrus manuscript have been found near Bakṣāli, the Mardan Tehsil, Peshwar district. On the East side of the Mardan and Bakṣāli road are some mounds, believed to be the ruins

of a former village, though nothing is known with any certainty regarding them, and it was while digging in a ruined stone enclosure on one of these mounds that the discovery was made. A triangular shaped stone 'divā' and a soapstone pencil, and a large loṭāh of black clay, with a perforated bottom, were found at the same place. Much of the manuscript was destroyed by the ignorant finder in taking it up from the spot where it lay between the stones ; and the remains are described as being like dry tinder, in some of the pages. However, the character which somewhat resembles 'Prakṛit' is clear and it is hoped that it may be deciphered when it reaches Lahore, whether we understand, it is shortly to be sent."

Subsequently, the manuscript was sent to Dr. Hoernle the head of the Calcutta schools, for examination and publication. Dr. Hoernle published some parts of it in the Indian Antiquary pages 33–48 and 275–279 Bombay 1888.

Later on the whole manuscript—as is the fate of almost every old Indian Manuscript—was taken to England and is now the property of the Bodlian Library Oxford. G.R. Kaye has edited this Bakṣāli manuscript in three parts, the first two parts are published in 1927 and the third part in 1933 on behalf of the Archaelogical Survey of India, New Imperial Series Vol. XLIII. But it would be observed that it is full of mistakes and Kaye's conclusions are always misleading.

The manuscript is written in Śāradā characterts on leaves of birch bark (birch tree is known as the bhūrja tree) and consists of some 70 leaves. Unfortunately much of it is extremely mutilated but whatever that remains has been put in order and translated in English and we observe that it throws an amount of light on the developments in Arithmetic and Algebra.

The most noteworthy feature about this manuscript is that it is clearly a copy of some old manuscript and the original copy must have been composed between 2nd century B. C. to the 2nd century A. D. The present find is but a copy of the original and the copying seems to have been made at about the 8th century A. D.

Following are the proofs to establish the statement.

(1) In Recto 1 before the 10th sūtra (rule) in the *Middle* of the page is a sentence

द्वितीयपत्रे विवरितास्ति ।.

This is explained on the second page; while the second page of the Copy deals with a different type of example.

(2) In Recto 3 before the 14th sūtra in the *middle* of the page the following sentence is found,

पत्रेभिलिखितास्थित चतुर्दश सूत्रम् ।. On the page (from which it is copied) it is numbered as the XIV formula. Hence it is numbered in the copy as well, as the XIV formula.

(3) Again in Verse 4 in the space between the 4th and the 5th line is written

सूत्रे भ्रान्तिमस्ति ।

meaning thereby that it is not clear to the scribe.

There are several such statements which show that the present has been a copy of an ancient work which also must have been written on 'leaves' (पत्र).

The language of the text is written in what is called Gāthā dialect and is rather an admixture of Sanskrit and Prakrit; and from the inscriptions of the Indo—Sythian Kings, it seems that this style was in vogue for literary purposes, till the end of the 3rd century A. D. So the original manuscript must have been written before 300 A. D., or it might be the still older composition in Sanskrit

that has been rendered in Gāthā dialect and written in the Śāradā script from which the present copy has been made. That it is a copy of an older work seems to have been connived at by, or escaped attention of Kaye and his pains therefore regarding the deductions from the state of the formats of the present birch bark manuscript are of no use in establishing the date—which he has conveniently and cleverly tried to bring some time in the 12th century by the considerations of the nature of the formats.

Another curious character of the manuscript is the symbol of operation for subtraction. The modern symbol for plus + has been used to denote the negative sign. Thus $\left| \begin{matrix} 12 & 7+ \\ 1 & 1 \end{matrix} \right|$ means $\frac{12}{1} - \frac{7}{1} = 5.$

The presence of this symbol has been a matter of much speculation and conjecture. Thibaut sought to establish its connection with the supposed Diophantine negative sign ⋔ (reversed ψ) tachygraphic abbreviation for λειψις meaning wanting. Kaye believed it, as he thereby could conveniently get one more opportunity to show the influence of the Mediterranean science over the Indian. But the Greek sign for minus in not ⋔—. It is rather arrow like ↑. And it cannot be definitely said that Diophatus ever used it.* On the suggestion from Prof. Thibaut, Dr. Hoernle observes the slight resemblance between the two signs but correctly states that 'considering that the Hindus did not get their elements of Arithmetical science from the Greeks, a native Indian origin of the negative sign seems more probable. Also the recent researches of Datt and Singh finally establish that the Indian advance in Mathematics and parti-

* See Introduction page 7.

cularly in Arithmetic was quite original and the Greeks are indebted to India for much of the developments in this science. As such Thibaut's contention is quite baseless and far from facts. Moreover the so called negative sign of Diophatus resembles more an arrow ↑ than the Bakṣāli cross + in which both the vertical and the horizontal lines bisect each other.

Dr. Hoernle however made a plausible suggestion. He opines that 'it is the abbreviation (*ka*) of the word 'kanīta' (कनीत) diminished or it represents the sign *nyū* an abbreviation of nyūna (न्यून) diminished. The letter for *ka* or *nyū* in the Aśoka inscription would very closely resemble a cross +'. That it is a tachygráphic abbreviation of some Sanskrit word, is beyond doubt as the word yu (यु) is used for yuta (युत = summation) bhā भा for bhāga (भाग = division) etc. But neither the word kanita or nyūna is found in the Bakṣāli text. The corresponding word used for subtraction is kṣaya (क्षय) and the first word *kṣa* (क्ष) in Bakṣāli or Brahmi characters differs from the simple cross (+) in having a little flourish at the lower end of the vertical line. Dutta rightly suggests that for convenience this flourish seems to have been dropped. Thus the mark + which is not met in any other work on arithmetic from Āryabhaṭṭa and Brahmagupta onwards [for they use a dot to denote a negative sign ($\dot{4}$ or $\overset{\circ}{4} = -4$)] definitely shows that it is a work of great antiquity at least it cannot be placed later than 300 A. D.

Another point which clarifies the question of date is the lavish use of the words dināra and drāma occurring as the denominations of money, in many examples. The way in which these terms have been used, show that the dināra and drāma formed the legal tender of the day. Dināra was a gold coin and drāma a silver one. The dināra

and drāma were current in India in the time of Indo-sythian kings who ruled from the first century B. C. to the third century A. D., as many of the coins (dināra and drāma) most numerously found in excavation are those of Kaniṣka and Haviṣka who reigned in the second century A. D. Curiously enough the word dināra occurs in Mahāvīr's works: in one sum the wages of a coolie work upto 18 dināras a day. On this score Kaye opines that it must be a copper coin and it cannot bear any relation to the gold dināras of the time of Indo-Sythian Kings. On this score the Bakṣāli has been placed in the twelfth century by Kaye. But a little thought shows that Kaye's arguments cannot be held true, for, in the first place we cannot place any reliance on the text book wages; secondly in Bakṣāli Manuscript (folio 60*) the day's wages are given from 1½ dināras to 3 dināras, these are not exhorbitant as to be unbelievable if we consider that India in those days was the richest country possible. If we do not want to accept the second contention why can we not believe that for the sake of a sum in fractions and the rule of three, such lavish wages might have been given as a practice in calculation? Hence the argument advanced by Kaye cannot be taken as authentic.

Again the modern method of division which has been given by Śrīdhar, Āryabhaṭṭa II and Mahāvīr is not at all found in the Bakṣāli, The text book where every operation has been elaborately done, the division also would have been shown elaborately. But the absence of this method would at least prove that the work cannot belong to a later date than Āryabhaṭṭa II (950 A. D.) much less to the 12th century.

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* In this example two, three are the wages of princes or vassals who attended a sovereign king and not of a coolie.

Again in the text itself there is no example, neither a sūtra nor an allusion to the operation of the cube root. The description of the operation of the cube root is found in 'Āryabhaṭṭīya' by Āryabhaṭṭa who lived in the 5th and 6th century. So if the Bakṣāli manuscript is written in the 12th century as Kaye believes, the manuscript that contains almost a variety of problems on Algebra and Arithmetic, ought to deal with a problem on cube-root.

Another argument of Kaye which he makes much of, is the following. "There are other indications rather than evidence of the age of this work in the material of the text. The occurrence of the square-root rule would not be an anachronism if it were found in any Indian text book—from the time of Āryabhaṭṭa onwards—but it occurs in no other Indian text until very late indeed.....it appears in the Bakṣāli Manuscript and is probably due to direct western influence, probably to Muslim influence." Kaye further adds with a pretension of impartiality, that "such evidence is valueless to the inexpert; it can only carry weight with those who have every knowledge of the Mathematical field of the period".

Kaye thereby wants to suggest that the above rule was invented at a later date by the westerners and through them or through the Muslims the Indians copied it. This square—root rule according to Kaye was not even noticed by the Hindus until the 16th century (see Kaye's B. MSS page 31). But it would be seen that the whole argument of Kaye toples down like a house of cards when we observe that this square-root rule was known to the Indians at least in 750 B. C. if not earlier and the composers of the Śulva-Sūtras have made a free use of the rule $\sqrt{A^2 + b} = A + \frac{b}{2A}$ approximately in obtaining the value of Saviśeṣa i. e. $\sqrt{2}$

(see the chapter III pages 39 and 40). In the face of it Kaye's arguments appear to be baseless and absolutely ridiculous. There is no wonder that the rule which was first given before 750 B. C. should be so extensively used and be further developed with the application of continued fractions in Bakṣāli Manuscript.

In the text proper there are many examples which contain references to God Śiva or Śūlin, God Vāsudev, Sītā, Rāvaṇa, Śatrudāma (Śatrughna), Pārtha or Arjuna, Udhiṣṭhir, Suras, Asuras etc. but they do not help us to fix the date of the work as these Gods or the personages are the heroes of Indian Mythology.

Regarding these non-mathematical elements Kaye has tried to make much of, in bringing the date of the work to a later period. For instance he gives the reference to folio 50 on which appears the following :

.... वशिष्ठ पुत्र
सिकस्यार्थे पुत्रपौत्र उपयाग्यें भवतु
लिखितं च्छाजकपुत्र गणकराजे ब्राह्मणेन ।
सर्वेषांमेव शास्त्रानां गणितं मूर्ध्नि तिष्ठति ।
अद्यावसाने संसारे उत्पन्न ——महत् ।
पश्चात्सृष्टि तदाकर्तु शिवेन परमात्मनः ।
...... याद्यंचमुत्पन्नं गणितं संख्याकरणम् ।

"From this it appears that the section of the text is written by a Brahmin the son of Chhājaka, (च्छाजक), possibly this name is Sajjka (सज्जक) which occurs several times in Rājataraṅgiṇī. Sajjaka was the Superintendent of the Śeda office in Kalhaṇa's time (XII century)". After connecting Chhājaka with Sajjaka (mark the difference between the names) Kaye himself adds that there is no justification for connecting this individual with the author of our text.

After piling up such ridiculous data he goes on to establish the date of the composition to the 12th century.

There are a number of such instances and it is not worthwhile to enumerate them and refute them as they are quite illogical. Almost all the western scholars like Kaye have started their researches with a view to establish a certain statement, convenient to prove their superiority in each and every branch of knowledge, and not with a pious impartial desire to know the facts, as such their arguments cannot be held authentic.

Hence taking into consideration the language, the references to the coins dināra and drāmma, the symbol + for the negative sign, absence of the method for finding the cube-root together with the internal evidence we can legitimately place the work in a period extending from 200 B. C. to 200 A. D.

Having established the point regarding the date let us enumerate the important additions Bakṣāli Manuscript has made to the development of Mathematics upto the 2nd century A. D.

The work contains a variety of problems. It deals with

(1) Examples of the rule of three and profit and loss.

(2) Examples on computation of gold.

(3) Solutions of linear equations with as many as five unknowns.

(4) The solution of the quadratic equation.

(5) The square-root approximations.

(6) The arithmetic progressions.

(7) The compound series.

(8) Quadratic indeterminate equations.

I The rule of three

(Recto 61)

The first three lines deal with an example on verso 60 carried overleaf. After this follows :—

†॥ उदा. ॥ ....

*(रा) जपुत्रौ द्वयौ केचिन्नृपति सेव्यसन्निवै ... मेकस्याहे द्वयोरस्य द्वितीयस्य दिवार्धकम्। प्रथमेन द्वितीयस्य दशादिनार ... त्तवान्। केन कालेन समतां गणयिता वदसुमे ॥ ...... ॥ 3 ॥ दत्तम् 10 ॥ करणम् । अहद्रव्य विशेषं च । तम् ...... 2 । 3 ।

	1	13	30	फ. (65)*	
	1	6	1		
दत्तम् जातम्	1	3	30	फ. 45	55 समधनजा............
55	1	2	1		

The letters or figures marked with ( )* are supplied as they are evident and wherever there are dots...... that portion is mutilated.

This example* is as follows: Two princes await upon a king (or they are the page boys of a king). For his services, one gets $2\frac{1}{6}$ a day and the other $1\frac{1}{2}$. The first has to give to the second 10. Calculate and tell me when they will have equal amount.

---

† This example has been given by Hoernle in the Indian Antiquary Bombay 1888 page 44. Look at his rendering.

"Two Rajputs are the servants of a king .... etc." Note down the difference between the words *Rājputra* and *Rajputa*. There is a world of difference between the two words. Rājputra means a prince and Rajputa is a resident of Rājputānā. Depending upon this mistake, Kaye in his work on Bakṣāli already referred to P. 19 line 40, tells that "Rajputs are described as servants of a king". Depending upon such foolish mistake, Kaye wants to suggest that in the 12th century Rajputs might be servants of a Mohommedan kings, and places the work in that century. The connection of Rājputra and Rajputa is as ridiculous as that of Chhājaka and Sajjaka referred to on P. 57.

The working of the example is quite clear.

The first gets $\frac{13}{6}$ dināras.

The second gets $\frac{3}{2}$ dināras.

Consider the denominators 6 and 2 together with the number 10 the first has to give to the other.

So the L. C. M. of 2, 6, 10 is 30.

So the 30 days is the answer. Then follows the verification.

By the rule of three

	day		days		dinā.	Dināras.
For	1		30	::	$\frac{13}{6}$	= 65
	day		days		dinā.	Dināras
For	1	:	30	::	$\frac{3}{2}$	= 45

$$65 - 10 = 55$$
$$45 + 10 = 55$$

First has 55. The second has 55. Thus they have equal quantities.

(Verso 61) सूत्रम् ॥ (विक्र)*येन क्रयं भाज्यं रूपहीनं पुनर्भजेत् ।
(लाभेन)* गुणयेत्तत्र नीवी भवति तत्रच ॥
उदा. ॥ द्विभिः क्रिणाति यस्सप्त विक्रिणाति त्रिभिषट् ।
अष्टादश भवेल्लाभ का (नीवी तत्र)*कथ्यताम् ॥

| 7 / 2 | 6 / 3 | 18 / 1 | लाभ | करणम् । —वि ।—

(Recto 62) .... गुणितं जातम्...... नीवी जात । ... सदात्य ......
.... न ॥ यदि द्विभिस्सप्त लभ्यते । तदाचतुर्विंशतिभिः किम्

| 2 / 1 | 7 / 1 | (24)* |

.... लमृ 84 ॥ अस्य विक्रयं क्रियते । यदि षड्भि । त्रय
.... ना लभ्यते तदा चतुराशातिभिः किम् | 6 / 1 | 3 / 1 | 84 / 1 | फलम् 42

मूल्म् 24 । पात्यशोणम् 18 एषलाभः (चौपञ्चसूत्रम् 54)

नीविः or नीवी means capital.

Translation of the sūtra :—Divide the (rate-of) purchase by the (rate-of) sale. Subtract one, again divide one (by the result obtained) and multiply by the profit and it becomes the capital.

i. e. if p is the rate of purchase
and if r is the rate of sale
P is the profit the rule states that the capital C is given by

$$C = \frac{1}{\left(\frac{p}{r}\right) - 1} \times P.$$

The example that follows illustrates this rule. Example given is as follows: One buys 7 for two and sells 6 for 3. If 18 is the profit what is the capital?

The solution with the help of the rule is given by

$$C = \left[\frac{1}{\frac{(\frac{7}{2})}{(\frac{6}{3})} - 1}\right] \times 18 = 24.$$

though this part is mutilated.

Then follows the verification. By the rule of three

For 2, the number bought is 7
For 24, ,, ,, ,, ,, 84
For 6 are sold 3
For 84 42
$42 - 24 = 18$ is the profit

Recto 62 (वि)*क्रयं भाजयेच्चैव गुणयेत् क्रमपिण्डताम् ।
रू(पोन)*मूल गुणयेत् लब्धं लाभं च प्राप्यते ॥

Then follows an example which is carried overleaf to Verso 62. This rule is for obtaining the profit. It states

$$P = C\left(\frac{P}{r} - 1\right)$$

First part contains the example carried overleaf from Recto 62. Then is given the rule

(Verso 62) विक्रयं भाजयेच्चैव गुणयेत् क्रमपिण्डवत् ।
विभक्तं सच कर्तव्यं गुणयेत् मिश्रकं बुधैः ॥
यल्लब्धं सा भवेन्मू(लंतच)छेषं लाभपिण्डताम् ॥

This rule is another form of the two rules stated above viz.

$$\frac{C+P}{\left(\frac{p}{r}\right)} = C$$

(C + P) is called the mixed quantity.

First part contains the example carried overleaf from 62 Verso. Then is given the sūtra viz.

(Recto 63) विक्रयं च विभक्तव्यं गुणितं क्रयराशिवत् ।
कृत्वारूपक्षयं चैव विभक्तम् मूल्माप्नुयात् ॥

If L is the loss the rule states $C = \dfrac{L}{1 - \left(\frac{p}{r}\right)}$

Thus in the examples on profit and loss the different sūtras are given. They are in fact the various forms of one and the same equation, viz.

$$C = \frac{P}{\frac{p}{r} - 1} \text{ or } \frac{L}{1 - \frac{p}{r}}$$

according as it is profit or loss.

On each rule two examples have been added which show that at the time of the Bakṣāli composition, the variety of problems on profit and loss were tackled with ease and accuracy.

II Examples on computation of gold

(Verso 16) It begins with a sūtra :

............संगुण्यं कनकस्त द्युतिर्भाजयेत्ततः ।

सम्युतैरेव कनकैरेकेकस्य क्षयो हि सः ॥ उदा. ॥

(एक द्वि त्रि चतुः)* संख्या सुवर्ण माषैकैक्कणैः। एक द्वि त्रि चतुः संख्या

.... रहिता समभागताम् । स्थापनां क्रियते । एषाम्

$\ldots + \left|\begin{array}{l}2\\2\end{array}\right. + \left|\begin{array}{l}3\\3\end{array}\right. + \left|\begin{array}{l}4\\4\end{array}\right. + \Big|$ करणं । क्षय संगुण्य कनकादि क्षयेन संगुण्य जातम् 1।4।9।16।

............ति । एषा युति ।30। कनकायुति ।10। अनेनभक्त्वा ल(भते ।3)*

This is but an example on averages. The rule gives that if x_1, x_2, x_3......are the quantities of kṣaya (i. e. impurity) in the pieces of gold weighing w_1, w_2, w_3.........then the average impurity $\bar{x}$ (i. e. the impurity in the mixture) is given by

$$\bar{x} = \frac{w_1 x_1 + w_2 x_2 + w_3 x_3 + \ldots\ldots\ldots\ldots}{w_1 + w_2 + w_3 + \ldots\ldots\ldots\ldots\ldots}$$

In the given example

$$x_1 = 1 , x_2 = 2 , x_3 = 3 , x_4 = 4$$

and $$w_1 = 1 , w_2 = 2 , w_3 = 3 , w_4 = 4$$

then $$\bar{x} = \frac{1 + 4 + 9 + 16}{1 + 2 + 3 + 4} = \frac{30}{10} = 3.$$

There are two more sūtras given, which are but the different forms of one and the same sūtra given above. Each sūtra follows with an example to illustrate the use.

III Linear equations

(Recto 1) After some sūtras, follows : ॥ उदा. ॥ पञ्चानां वणिजामध्ये विक्रीयते किल । त्वत्रोक्तं माणिमूल्यं कियद् भवेत् .... दम् । अर्धत्रिभाग पादांशापञ्च मा (षड्विभागच)*

(Verso 1) .... । । । ।

.... स्यन्ततो प्रोझ्याः षाड्संक्रियत ।

जाता	120	90	80	75	72	तत्र प्रोझ्या जातम् । 120 । 0
एषयोगक्रि	60	60	60	60	60	ते जाता 437 अतो स ...

...... शेषम् 377 एष माणिमूल्यम् । चतुर्णां सख्य सर्वस्वम् ॥ प्रथमस्य संख्यार्धम् (।60)* । (90 । 8)* 0 । 75 । 72 । चतुर्णायोग 317 प्रथमार्धेन षष्टिभिर्युतम् 377 (प्रथम)* तृतीय-चतुर्थपञ्चमस्य धनं सर्वस्वं 347 द्वितीयत्रिभागं । 30 । ...... युतं 377 एष द्वितीयस्य धनं भवति । पुनर्प्रथमाद्वितीयचतुर्थपञ्च (मस्यधनसर्वे)* स्वं 357 तृतीयस्य पादं । 20 । एष युतं 377 एष तृतीयस्य । पुनरपि प्रथम द्वितीय तृतीय पञ्चस्य * (362 चतु-र्थस्य पञ्चभागं 15 एषयुतं 377 एषचतुर्थस्य)*

(Recto 2)

स्य धनं भवति ॥ अथ पृथ
स्य पष्टि शेषं 377 ॥ अथ द्वितीय | 75 | स्य | 120 30 | एवं संक्रियेते | | 120
द्वितीयस्य भवति । अथ तृतीय | 72 | स्य | 80 75
...377 तृतीयस्य धनं भवति । चतुर्थस्याक्रिया | 72 | ते | 120 | 9
एवं 377 चतुर्थस्य धनं भवति । पञ्चमस्य क्रिंयतेस्या | 90 | 2
पणं | 120 | (एवं पञ्चमस्य 377 । एषमणिमूल्यं पृ......)* | 80 | 7
90 | | 15 | 7
80
75
72

Then follows another example which is carried overleaf. The statement of the example is mutilated but the proofs given help us to know the wording. It is as follows:—

Five merchants together, want to buy a jewel. Half the capital of the first together with those of the others, one-third of the capital of the second together with those of the others, one-fourth of the capital of the third together with those of the others, one-fifth of the capital of the fourth together with those of the other, one-sixth of the capital of the fifth together with those of others is

the price of the jewel, find the price of the jewel and the capital of each merchant.

Let x_1, x_2, x_3, x_4, and x_5 be the capitals of each merchant, respectively and p the price of the jewel. Then

$$\left.\begin{aligned} \frac{x_1}{2} + x_2 + x_3 + x_4 + x_5 = p \\ x_1 + \frac{x_2}{3} + x_3 + x_4 + x_5 = p \\ x_1 + x_2 + \frac{x_3}{4} + x_4 + x_5 = p \\ x_1 + x_2 + x_3 + \frac{x_4}{5} + x_5 = p \\ x_1 + x_2 + x_3 + x_4 + \frac{x_5}{6} = p \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots(A)$$

These equations give

$$\frac{x_1}{2} = \frac{2}{3}x_2 = \frac{3}{4}x_3 = \frac{4}{5}x_4 = \frac{5}{6}x_5 = q \text{ say.}$$

$\therefore$ The equations (A) become $\frac{377}{60} q = p$.

Then it would be observed that the integral value was required. So we find that q is taken equal to 60, and we get $p - 377$. Even if we take q equal to any multiple of 60 a number of answers are obtained. This is in fact, the origin of the indeterminate equations of the first order which are of the type $\frac{ax}{c} = y$ and that is why this particular type of theory developed a great deal and was later on perfected by Bhāskarācārya in the 13th century 400 years before it was discovered in Europe.

Kaye's rendering of this example is quite wrong. He says that the example appears to have been somewhat as follows: Combined capitals of five merchants less half that of the first, one third that of the second, one fourth that of the third, one fifth that of the fourth, or sixth that of the fifth is the price of the jewel. Find the cost of the jewel and the capital of each merchant, (See page 168 of his B. MSS. part III).

Having taken $p = 377$, the values of x_1, x_2, x_3, x_4, x_5 are 120, 90, 80, 75 and 72 respectively.

The verification of the example is given on Recto 2 as well.

(Verso 3) ...स्य हयान्नवः । उष्ट्रादश तृतीयस्य ....

प्रदत्तं च परस्परम् । पृथग्धनं तु वणिजां मूल्यं वा प्राणिनां पृथक् ।
यद्द ... वक्तुं ततो मे च्छिन्द्धिः संशयः । । 7 अ । 9 हय । 10 उष्ट्र ।
वणिजका 3 य ... णिक् पिण्डहतम् । पिण्ड । 7 । 9 । 10 । देयं 3
शुद्धशेषम् । 4 । 6 । 7 । (ततः)* परस्परकृतं गुणितं जातम्
। 168 । 168 । 168 । शेषेनतु वि ...... | 168/4 | 168/6 | 168/7 |
लब्धं । 42 । 28 । 24 । एष प्रत्येक मूल्यमेकैकस्य गुणितजातानि
अश्वैः हयैः उष्ट्रेभ्यः । 294 । 252 । 240 । एकैक ... ... जाता
। 262 । 262 । 262 । एते समधनजाता

The general meaning of the words Aśva (अश्व) and Haya (हय) is a horse. Horses are said to have seven breeds. For

अमृताद्वाष्पतो वह्नेर्वेदेभ्योऽडाच्चगर्भतः ।
साम्नोहयानामुत्पत्तिः सप्तधा परिकीर्तिता ॥

Thus *haya* and *aśva* are two different breeds of the horse. So there is no repetition as Kaye takes it to be. Dr. Hœrnle takes the meaning of *haya* as 'yak' which has no basis whatsoever.

The example can be constructed from the mutilated sheet as follows:—One possesses 7 aśvas, the second 9 hayas and the third 10 camels. Each gives two animals one, to each of the others. Find the price of each kind of animal or the property of each.

If x_1, x_2, x_3, are the prices of each of aśva, haya and camel respectively, then

$$\begin{aligned} 5x_1 + x_2 + x_3 &= p \\ x_1 + 7x_2 + x_3 &= p \\ x_1 + x_2 + 8x_3 &= p \end{aligned}$$

Whence subtracting one from the other we get

$$4x_1 = 6x_2 = 7x_3$$

Now $4 \times 6 \times 7 = 168$ is taken as the L. C. M.

$\therefore \quad x_1 = 42, \ x_2 = 28, \ x_3 = 24.$

$\therefore$ Their respective possessions amount to

$$42 \times 7 = 294, \ 28 \times 9 = 252, \ 10 \times 24 = 240.$$

Hence it would be observed that without the value of p, they used to solve their problems of simultaneous equations in such a way that the answers are complete integers. If we take any multiple of 168, we would have another set of answers. The L. C. M. 84 would have given another set of answers.

From the fragments of verso 39 recto 39 and verso 40 it would be seen that one example illustrated the use of the equations

$$x\,(x + y + z) = 60$$
$$y\,(x + y + z) = 75$$
$$z\,(x + y + z) = 90.$$

The values of x, y, z, are 4, 5, and 6 respectively.

(IV. V. VI.)

There are a number of examples which at one and the same time involve the theory of arithmetic progressions, the solution of the quadratic equation and the method of extracting the square-root with approximations. Hence the illustrations of such examples would be under the same head.

Here is a problem on arithmetic progression.

(Recto 9) .... र ....स्ठ .... च ॥ उदा. ॥

.... माप्त द्वि जन्मभिः । तत्पुनस्ते समंभत्क्वाद ....

...स्स माप्तवान् संख्यायाः कतिमच्चक्षुः कति विप्रा कति ... र ... म्

आ. $\frac{1}{1}$ | उ. $\frac{1}{1}$ | फ. $\frac{0}{1}$ | लब्धं $\frac{10}{1}$ | करणं ॥ लब्धं द्विगुणितं कृत्वा तत्र
लब्धं । 10 । द्विगुण । 20 । तथाद्युनां | $\frac{18}{1}$

......विभाजितं । अत्रोत्तर । 1 । अनेन भक्त्वा जातम् तद्दे......
...... कम् | $\frac{19}{1}$ | अयं प्रश्नाप्राभुणा एकोनविंशाति । स्थाप ......
.... | आ. $\frac{1}{1}$ | उ. $\frac{1}{1}$ | फ. $\frac{19}{1}$ | रूपोणा करणेन फलम् ।1।

आदि means the first term.

उत्तर means the common difference.

The example seems to be like this: Find the value of n the number of terms of an A. P. whose first term is 1 and the common difference is also 1 if the sum is $10n$. It states

$$10n = \left[(n-1)\frac{1}{2} + 1\right] n \quad \therefore \quad n = 19.$$

The method given in the text is this: Multiply 10 by 2 .and subtract double the first term and then add one i. e. $[(10 \times 2 - 2) + 1] = 19$.

(Verso 7) ....
...प्रतान्द्न | आ. $\frac{3}{1}$ | उ. $\frac{4}{1}$ | फ. $\frac{0}{1}$ | नित्यदत्त $\frac{7}{1}$ | आदि विशोध्य
। 3 । नियतम् । 7 । विशोध्य । 4 । उत्तरार्धेन भाजितम् । उत्तरम्
। 4 । अ ...... जितम् | $\frac{4}{2}$ | जातम् । 2 । लब्धं सरूप एष रूपाधिकं
। 3 । एषकाल .... | $\frac{3}{1}$ | उ. $\frac{4}{1}$ | प. $\frac{3}{1}$ | रूपोणा करणेन
फलं रू. । 2 । ॥

After this the method is verified by the rule of three.

If $a = 3$, d the common difference $= 4$, obtain the value of n the number of terms if the sum is $7n$.

$$7n = \left[(n-1)\frac{4}{2} + 3\right] n$$

$$\therefore \quad n = 3 \qquad \therefore \text{ the sum is } 21.$$

Thence comes the deduction from the formula

$$s = n\left((n-1)\frac{d}{2} + a\right).$$

From this rule the value of n is obtained and though the rule seems to be mutilated, its use in the text is extensive. From it the value of n is given as

$$n = \frac{\sqrt{(2a-d)^2+8ds}-(2a-d)}{2d}.$$

It would be observed that this is nothing but the solution of the quadratic equation in n given by

$$s = n\left[\,(n-1)\,\frac{d}{2}+a\,\right]$$

i. e. $dn^2 + (2a-d)\,n - 2s = 0$

$$\therefore \quad n = \frac{-(2a-d) \pm \sqrt{(2a-d)^2+8ds}}{2d}$$

The negative sign for the radical is omitted: and hence the value of n is given. The use of this sūtra can be seen from the following example.

(Verso 65) ....

च्यते | आ. 1 / 1 | 3 ......6 | अष्टोत्तरे गुणिते । अष्टघ्नम् । 480 ।

उत्तरघ्ने ... व द्विगुणमादि । आदिद्विगुण । 2 । च योञ्झितम् । च उत्तरम् । अतो उत्तरपातयित्वा एकं भवति । 1 । वर ... | 21
...... निक्षिप्य धनस्य । 481 । मूलम् श्लिष्टकरण्य .... | 40
... वशम् | 882 / 40 / 42 | शेषं चत्वारिंश पृथक् स्थाप्य । 40 । | 42

थोजस् | 42 | | 922 / 42 | तन्मूलवर्जितस् ॥ तन्मूलम् । नोत्तरा ... ... अं 880

It will be observed the whole example is mutilated, but by looking to the method in which it is worked out, we can supply the data. Hear

$a = 1, \quad d = 1$ and $s = 60.$

$$\therefore \quad n = \frac{\sqrt{(2a-d)^2+8ds}-(2a-d)}{2d} \text{ gives}$$

$$n = \frac{\sqrt{1+480}-1}{2} = \frac{\sqrt{481}-1}{2}$$

In order to find the square-root of 481 the rule that has been used is

$$\sqrt{A^2 + b} = A + \frac{b}{2A}$$

approximately, the method that has been used by the composers of the Śulvasūtras.

$$\sqrt{481} = \sqrt{441 + 40}$$

$$= 21 + \frac{40}{42} = 21\frac{40}{42} \quad \text{first approximation.}$$

$$\text{But} \quad 21\frac{40}{42} = \frac{882 + 40}{42} = \frac{922}{42}.$$

This seems to be the result. Kaye has tried to obtain the second approximation by the same rule and the connection he has shown of this and the Recto 56 is worth noting.

(Verso 57) .... र ....
ऽष्टोत्तरघ्ने गुणिते 40 द्विघ्नमादि च या ....
.... निक्षिप्य । 41 । मूलम् | 6 / 5 / 6 | शेषच्छेदोद्विसंगुणी
शुद्धस्तस्मात् अकृति श्लिष्ट कृत्यूना शेषच्छेदो द्वि ....
...... दल संश्लिष्ट हृति कृतिक्षयः अकृति श्लिष्ट ....

...... दा द्विसंगुणकृत	6	तद्वर्ग	6			दल
	5		5	25		
	12	12	12	144		

5	11833	हृ	......	184
8	1848		1848	

कृतिक्ष कृतम् एष मूलम् ॥ तन्मूलम्
(Recto 57) ...... न्मूलमेकम् । 1 । एष सदृशे पतित जाता | 9985 / 1848 |
...... लंभक्तमुत्तरं द्विगुणं 2 अनेन भा
(9)* 985 | एष पञ्चकस्य पदम् ॥ ....॥ अस्यप्र
(36)* 96 | सूत्रम् ॥ एकोनराशि द्विधा स्थाप्य उपासे......

It seems that the page in the text was kept reverse and therefore the example from Verso 57 is completed on

the Recto 57. Kaye has restored the example correctly. He gives the example as $a = 1$, $d = 1$, $s = 5$. Hence the value of n is given by the equation.

$$n = \frac{\sqrt{(2a-d)^2 + 8ds} - (2a-d)}{2d}$$

$$= \frac{\sqrt{41} - 1}{2}$$

Again to obtain the $\sqrt{41}$ the rule for approximation,

viz. $\sqrt{A^2 + b} = A + \frac{b}{2A}$ approximately has been made use of. $\sqrt{41} = \sqrt{36+5} = 6 + \frac{5}{2 \cdot 6} = 6\frac{5}{12}$

The second approximation is given by

$$\sqrt{41} = \sqrt{36+5}$$

$$= 6 + \cfrac{5}{12 + \cfrac{5}{12 + \cfrac{5}{12}}}$$

$$= 6 + \cfrac{5}{12 + \cfrac{60}{149}}$$

$$= 6 + \frac{745}{1848}$$

$$= \frac{11088 + 745}{1848}$$

$$= \frac{11833}{1848}$$

And $\frac{11833}{1848}$ is the very fraction given in the text.

But

$$n = \frac{1}{2}(\sqrt{41}-1)$$
$$= \frac{1}{2}\left(\frac{11833}{1848}-1\right)$$
$$= \frac{1}{2}\left(\frac{9985}{1848}\right)$$
$$= \frac{9985}{3696}.$$

The second approximation that is given above, is based upon the same rule. Thus

$$\sqrt{A^2+b} = A + \frac{b}{2A} \text{ — as the first approximation}$$

and $\sqrt{A^2+b} = A + \cfrac{b}{2A + \cfrac{b}{2A + \cfrac{b}{2A}}} = A + \frac{4A^2b+b^2}{8A^3+4Ab}$

is taken as the second approximation though it should be known as the third approximation. Here the second approximation ought to be given by

$$\sqrt{A^2+b} = A + \cfrac{b}{2A + \cfrac{b}{2A}} = A + \frac{2Ab}{4A^2+b}$$

But the application of the very rule for the second time, gave a greater degree of accuracy. Also note down the use of continued fractions. This is a noteworthy development. The author would have applied the method again and again but as it illustrates the very rule, it was not necessary.

VII

The Compound Series

Besides the series in A. P., there are compound series as well. Though the leaf on which these series occur, are

broken into two, still from scraps, the data can be collected and the example can be restored.

First part of the scrap deals with an example carried overleaf. Then follows the example:—

(Recto 51) कश्चित्राजा ददे दानं सप्तपञ्चाशकं बुधः। पञ्चा—प्रवक्षाम्यनुपूर्वशः।
द्विगुणाद्विगुणं चैव रूपरूपोत्तरै—प्रथमे प्राप्तं किं प्राप्तमपरे जने ॥

There are other small scraps but it seems that they fail to give any working of the example. The example can be formed inspite of the lost parts.

" A certain king gives 57 dināras (say) to five wise men. First of all he gives a certain amount to the first man and subsequently, each time doubles the amount he gives to the previous one. On seeing that he has left with him, some more dināras, he gives to the first, whatever he gave to the first four on the previous occasion, to the second, whatever he gave to the first three, to the third, whatever he gave to the first two, and to the fourth, whatever he gave to the first previously and to the fifth he could not give anything, as he had exhausted all the 57 dināras. Find what the first got and the others".

The working of the example in the Bakṣāli fashion is as follows:—

1	2	3	4	5	
x	$2x$	$4x$	$8x$	$16x$	
$15x$	$7x$	$3x$	x	0	57
$16x$	$9x$	$7x$	$9x$	$16x$	$57x$ Total

$$\therefore \quad 57x = 57 \quad \therefore\ x = 1.$$

This represents the series

$$\left. \begin{array}{l} u_1 + 2u_1 + 2^2u_1 + 2^3u_1 + 2^4u_1 \\ \quad + [\, S_4 + S_3 + S_2 + S_1 \,] \end{array} \right\} = 57$$

where S_r denotes the sum of the series

$$u_1 + 2u_1 + 2^2u_1 + 2^3u_1 + 2^4u_1 \text{ to } r \text{ terms}$$
$$\text{where } r = 1, 2, 3, 4$$

∴ Their respective gifts are 16, 9, 7, 9, 16.

(Verso 51) । 27 । 81 । । करणम् । उत्तर———ज्य

तत्रोत्तर राशीनां योग 87

एषधना दृश्या शोधनीया जाता 24(2)*

... जाता । पुरुषा ॥ ...... । 9 । 27 । 8 । योग । 2 । अने

जाता । 2 । एष द्वौ प्रथमस्य धनम् ॥ । 2 । 6 । 18 । 54 । 162

उत्तर राशियुतम् जातम् । $\left| \begin{array}{c} 2 \\ 1 \end{array} \right\| \begin{array}{c} 15 \\ 2 \end{array} \left\| \begin{array}{c} 48 \\ 2 \end{array} \right\| \begin{array}{c} 147 \\ 2 \end{array} \left\| \begin{array}{c} 444 \\ 2 \end{array} \right\|$ एषाम्

Then follows another example.

This too, is the illustration of the compound series. Though much of it is lost, still from whatever figures we get, we can be definite about the result obtained. It is something like this:

$$\left. \begin{array}{l} u_1 + 3u_1 + 3^2u_1 + 3^3u_1 + 3^4u_1 \\ \quad + \frac{8}{4} [\, S_1 + S_2 + S_3 + S_4 \,] \end{array} \right\} = 329$$

now $\frac{3}{4} [\, S_1 + S_2 + S_3 + S_4 \,] = 87$ is given where S_r is the sum of the r terms of the series

$$u_1 + 3u_1 + 3^2u_1 + 3^3u_1 + \dots\dots\dots$$

which we denote by $u_1 + u_2 + u_3 + u_4 + \dots\dots$

$$1 + 3 + 9 + 29 + 81 = 121$$
$$329 - 87 = 242$$

$$\text{and since } \frac{242}{121} = 2 \quad \therefore \quad u_1 = 2$$

$$\text{Then } S_1 = 2, S_2 = 2 + 6, S_3 = 2 + 6 + 18 = 26$$
$$\text{and } S_4 = 2 + 6 + 18 + 54 = 80$$
$$u_1 = 2$$

$$u_2 + \frac{3}{4}S_1 = 6 + \frac{3}{2} = \frac{15}{2}$$

$$u_3 + \frac{3}{4}S_2 = 18 + \frac{12}{2} = \frac{48}{2}$$

$$u_4 + \frac{3}{4}S_3 = 54 + \frac{39}{2} = \frac{147}{2}$$

$$u_5 + \frac{3}{4}S_3 = 162 + \frac{120}{2} = \frac{444}{2}$$

These are the very figures seen on the scraps and hence we conclude that the compound series have been efficiently solved.

VIII

The Indeterminate Equations

(Recto 59) ...... उदा ॥ कोराशिपञ्चमुता ——— उ ——— साराशिसम

मूलद कोराशिरिति प्रश्नः— | 0 / 1 | 5यु . मू . 0 / 1 1 | सा 0 7 + मू . 0 / 1 1 |

...... णाम् । युतहीनं चमेकत्वम् । 12 । ता

...... लम् ।6। द्विहृणाम् ।4। दलम् ।2। वर्गम् ।4। हीनेयुतिंच

कर्तव्या ।7+ । अनेन युति ।11। एषसाराषि ॥ अस्यप्रत्ययानयने

.... 11 यु 5 म् 4 / 1 1 1 | 11 7 + म् 2 / 1 1 1 | पञ्चादशसूत्रम् / 50

॥ सूत्रम् । गवां विशेषकर्तव्य धनं चैवपुन ....

The example is as follows :—Find the number such that when 5 is added to it or 7 is subtracted, the results are complete squares.

If x is that number, we have to solve

$$x^2 + 5 = a^2$$

$$x^2 - 7 = b^2.$$

The steps are quite clear. $5+7=12.$ $\frac{12}{2} = 6$

$6-2=4.$ $\frac{4}{2} = 2.$ $2^2=4.$ $4+7=11.$

$\therefore \quad x = 11$ for $11+5=16=4^2$ and $11-7=4=2^2$.
Unfortunately the general rule of solving such types of examples is not preserved.

It is not possible to deal with all the examples given in the Bakṣāli Manuscript with the probable explanations though it would be interesting to do so, as many of the examples can be supplied from the fragmentary data. With the help of the facsimiles of the Manuscript published by the Archāeological Department, Government of India and edited by Kaye, an attempt has been made above, to bring out the developments in Mathematics during the time of the original composition of the Bakṣāli Manuscript i. e. from 2nd century B. C. to the end of the 2nd century A. D. By printing the facsimiles of the text Kaye has rendered a service to the cause of research but his conclusions have been always found to be far from the truth.

If we are to tabulate the developments in Mathematics in India upto the end of the 2nd century, we will have to note the following :—

(1) The free use of the rule of three.

If the Mathematics of the Vedic age was concerned with the religious and the astronomical purposes, the Mathematics of this period became more practical and was used in the daily workings of the society, as is evident from the rules and the examples on profit and loss, on computation of gold, on wages of the servants, on alms to the poor &c. &c.

(2) The rule for finding the square-root of any number to any degree of approximations.

If the Sūtrakāras have applied the rule of approximations to find the value of $\sqrt{2}$, they called ' Saviśeṣa ', at the time of the Bakṣāli

composition the rule was made more elaborate and was utilised to obtain the square root of any number to any degree of accuracy by taking approximations in succession. In this as has been pointed out, the approach to the use of continued fractions was made.

(3) The solution of the quadratic equations. Upto that time it was a development of remarkable quality. The extensive use of it shows that its solution must be even older than the actual composition of the Bakṣāli.

(4) The theory of Arithmetic progressions together with that of the compound series marks a high degree of progress from the time of the Vedic age.

(5) It marks the beginning of the solution of the indeterminate equation of the first degree and of the second degree. Though the example on the indeterminate equation of the second degree is quite simple, still it marks a new era in that field, at the same time it acquaints us with the trend of the mind of the Hindu Mathematician.

Unfortunately all these developments were lost to the later Mathematicians and as has been pointed out, the Mathematicians from Āryabhaṭṭa I onwards, had to start the study of Mathematics again and once more sow the seeds. It might be due to the imperfect channels of communications and the vastness of the country, and the want of people to memorise the brilliant results and understand them (for in comparison they must have found Nyāya Mīmāṁsā and the other Śāstras easily digestible) still the fact remains that generations after generations lost touch

with the exceptional achievements of the past and whatever was of practical use, remained in the daily affairs of men. From such a deep and dire degeneration there arose again a galaxy of Mathematicians headed by Āryabhaṭṭa I and once more the tide turned. From Āryabhaṭṭa I onwards, we find a train of Mathematicians whose works were not only preserved but were studied for generations together to the end of the 16th century. Had there been a connecting link in the Vedic age Bakṣāli period and the mediaeval time, the Indian Mathematicians would have led the whole world in the field of Mathematics perhaps to this day.

CHAPTER V

MATHEMATICS OF THE MEDIÆVAL AGE.

SECTION I

Āryabhaṭṭa I

From Āryabhaṭṭa I begins the revival of the science of Mathematics in India. Old developments of the vedic age or the Bakṣāli period were in essence buried as it were through want of communications and the absence of writing material, but whatever practical Mathematics was there, remained in vogue, as it helped the people in their daily affairs. It was Āryabhaṭṭa who stands as the pioneer of this revival. His fame rests on a work entitled "*Āryabhaṭiya*" in which only one chapter deals with Mathematics, the other four being devoted to Astronomy. This chapter contains only thirty three stanzas and from the nature of the results derived, it is evident that he must have worked a lot to arrive at them, as almost all are the finished products—i. e. in the form of formulæ. The idea of putting everything in the verse form, must have presented a strong difficulty to him and therefore the voluminous work done by him has been made as compact as possible. In such a small chapter he deals with Arithmetic, Algebra and Geometry. Here are some of the most important results given by him.

(1) Area of a Triangle

GEOMETRY

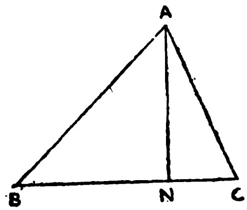
त्रिभुजस्य फलं शरीरं समदलकोटी भुजार्धसंवर्गः।

This gives the formula for the area of any triangle. Cajori says, "Āryabhaṭṭa gives the area of a triangle that

is true for only an isosceles triangle". This is an erroneous statement, either Cajori is misinformed or if he has translated this line his translation is wrong. Just read the explanation of this line.

Explanation :—त्रिभुजस्य क्षेत्रस्य या समदलकोटी लम्ब इत्यर्थः ।
लम्बस्योभय पार्श्वगते ये त्रिभुजदले त्रिकोणरूपे तयोरयं
लम्बएकएव कोटीर्भवति । तस्मात् दलकोटीत्युच्यते.

The word सम means *equal* as well as *common*. The latter meaning of the word must be taken here. If ABC is a



triangle and AN is perpendicular to BC. Then AN is called "koṭi" (कोटी) or the altitude of the triangle. By drawing this altitude the triangle is divided into two triangles having the common (सम) altitude. BC is the base or "bhujā" (भुजा). अर्धभुजा is half the base. समदलकोटी means the common altitude. संवर्ग means the multiplication. Hence the area of a triangle is one half the base into altitude (common to the two triangles). The meaning of समदलकोटी is not the altitude of two equal triangles. Such mistakes are often made by Western scholars: they are alien to the language and so cannot understand different shades of meaning or it might be that they get these works translated for them by local Pundits, that know little of Mathematics. It would be observed that Colebrooke has translated सूचि as needle and सूक्ष्म as neat. No doubt सूचि means a needle and सूक्ष्म means neat. But they forget the fact that words can have other meanings too. सूचि means a *cone* or a *pyramid* and सूक्ष्म means correct.

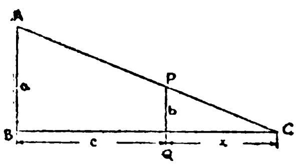
(as opposed to approximate). But such are his renderings of these words when Colebrooke translates the 44th verse on volumes given by Brahmagupta. Such instances can be piled up in numbers. So we cannot but conclude that the Westerners, because they are foreign to the Sanskrit language, are incapable of giving correct rendering; and hence such ridiculous mistakes prop up.

(2) The Similar Triangles

Here is an illustration of the theorem on similarity of triangles—i. e. the corresponding sides of similar triangles are proportional.

शङ्कुगणं शङ्कुभुजाविवरं शङ्कुभुजयोर्विशेषहृतम्
यल्लब्धं सा छाया ज्ञेया शङ्कोस्स्वमूलादि ॥

Explanation :—भुजा—दीपयष्टि (lamp-post) । तयोः विवरं अन्तरालभूमिः । ताम् शङ्कुना शङ्कून्नतिमानेन निहत्य शङ्कुभुजयोर्विशेषेण शङ्कून्नतिहीन दीपोन्नत्या विभजेत् । तम् लब्धं तस्य शङ्कोः छाया भवति ।



The verse gives us the length of the shadow CQ of the pole PQ. Here AB is the lamp-post (दिपयष्टि) or भुजा. BQ is the distance between the lamp-post and the pole PQ, i. e. विवर. Then the shadow is given by

$$x = \frac{bc}{a-b}$$

i. e. $$\frac{x}{b} = \frac{c}{a-b}.$$

which is the property of similar triangles.

Mahāvīr gives two forms of this equation in his work on shadows of gnomon.

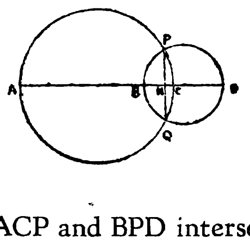
शङ्कूनित दीपोन्नतिराप्ता शङ्कुप्रमाणेन ।
तल्लब्धहृतो शङ्कोः प्रदीप शङ्कुतरं छाया ॥
शङ्कूनित दीपोन्नतिराप्ता शङ्कुप्रमाणेन ।
तल्लब्धहृताछाया शङ्कुप्रदीपमध्यं स्यात् ॥

छायाव्यवहार गणितसारसंग्रह
by
महावीराचार्य,

(3) $\rho = \frac{y^2}{2x}$

Here is a formula that seems to be the deduction from the theorem: "rectangles contained by the segments of chords of a circle are equal".

ग्रासोनेद्वे वृत्ते ग्रासगुणे भाजयेत्पृथक्त्वेन ।
ग्रासोनयोगभक्ते संपात शरौ परस्परतः ॥

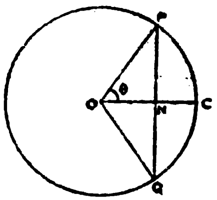


If two circles ACP and BPD intersect the part BC is called ग्रास, BN is the शर (arrow) of the arc PBQ and NC is the शर (arrow) of the arc PCQ

Let AC$=2a$, BD$=2b$, BN$=u$, CN$=v$ $\therefore$ BC $= u + v$ is the ग्रास. Then the above formula states that

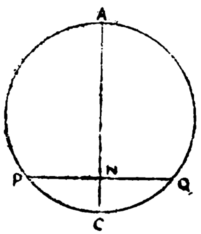
$$[\,2a - (u+v)\,]\,v = [\,2b - (u+v)\,]\,u.$$

It will be observed that the arrow of any arc of a circle whose radius is taken as unity is the versine of half the angle subtended by that arc at its centre.



$$\begin{aligned} CN &= OC - ON = OC - OP\cos\theta \\ &= OC - OC\cos\theta \\ &= OC\,(1 - \cos\theta) \\ &= OC \text{ versin } \theta \\ &= \text{versin } \theta \text{ .... if } OC = 1 \end{aligned}$$

The values of versines were often required in Astronomy. Hence the above formula has been obtained. Moreover the equation



$$[\,2a - (u + v)\,]\,v = [\,2b - (u + v)\,]\,u$$

gives $(2a - v)\,v = (2b - u)\,u$.

From the above figure we see that this equation gives

$$AN.\,CN = DN.\,BN$$

and each of the members is equal to PN^2.

Thus Āryabhaṭṭa knew the theorem i. e. $AN.\,CN = PN^2$

In other words

$$\frac{PN^2}{CN} = AN \text{ or } \frac{PN^2}{2CN} = \frac{AN}{2}$$

$$\text{If } PN = y,\ CN = x \text{ then } \frac{AN}{2} = \frac{y^2}{2x}$$

the very formula Newton made use of, to obtain the value of ρ the radius of curvature at the origin. It would be observed that this very theorem has been given rather elaborately by Brahmagupta. He states it as

इष्टशरद्वयभक्ते ज्यार्धकृति शरयुते कले व्यासौ ।
शरयोः फलयोरैक्यं ग्रासो ग्रासोनमैक्यं तत् ॥

Brahmagupta siddhānta.

(4) The Value of π

The value of π given by Āryabhaṭṭa is correct to four places of decimals. He states it thus:

चतुरधिकं शतमष्टगुणं द्वाषष्टिस्तथा सहस्राणाम् ।
अयुतद्वय विष्कंभस्यासन्नो वृत्तपरिणाहः ॥

आयुत means 10000

आसन्न means approximate

परिणाह means circumference

विष्कंभ means diameter.

It states $\pi = \frac{62832}{20000} = 3{\cdot}1416.$

Mark the word आसन्न which means approximate. So Aryabhaṭṭa knew that this value of π too, (which is correct to four places of decimals) is approximate. This shows that he expected his successors to try to obtain still more accurate value of π.

(5) The Area of a Circle

The area of a circle is given by the first line of stanza No. 7, which is

समपरिणाहस्यार्धं विष्कंभार्धहतमेव वृत्तफलम्

Area of a circle $= \frac{1}{2}$ (the circumference) $\times \frac{1}{2}$ (the diameter)

$= \frac{1}{2}(2\pi r) \times \frac{1}{2}(2r)$ where r is the radius of the circle.

$= \pi r^2.$

Algebra & Arithmetic.

(1)

The rule of three is stated as follows:

त्रैराशिक फलराशिं तमथेच्छाराशिना हतं कृत्वा ।
लब्धं प्रमाणभजितं तस्मादिच्छा फलमिदं स्यात् ॥

This is a definite improvement on Bakṣāli. The three parts in the rule of three have been defined as (1) प्रमाण = argument. (2) फल – fruit. (3) इच्छा – requisition. (4) इच्छाफल – produce. i. e. 1 and 3 are alike and 2 and 4 are alike.

The above verse states that the product of fruit and requisition divided by argument gives the produce.

(2)

Here is a rule for solving examples on interest.

मूलफलं सफलं कालमूल गुणमर्धमूलकृति युक्तम् ।
मूलं मूलार्धोनं कालहृतं स्यात्स्वमूलफलम् ॥

Compare this rule with that of Brahmagupta,

कालप्रमाणघातः परकालहृतो द्विघाऽद्यामिश्रवधात् ।
अन्यार्धकृतियुतात् पदमन्यार्धोनं प्रमाणफलम् ॥

Gaṇitādhyāya by Brahmagupta.

Both the rules state one and the same fact.

If P is the principle

i the interest for a period m

n the given period

a the amount; then the rule states

$$\sqrt{\left(\frac{Pn}{2m}\right)^2 + \frac{Pna}{m}} - \frac{Pn}{2m} = i$$

$$\text{or} \quad \frac{Pn}{2m}\sqrt{1 + \frac{4ma}{Pn}} - \frac{Pn}{2m} = i.$$

It would be observed that the result has been obtained by the solution of a quadratic equation. It should be noted that to make the example practical, it is essential that $\left(1+\frac{4ma}{Pn}\right)$ should be a complete square.

(3)

The method of inversion has been stated as

गुणकाराः भागहराः भागहराः ये भवन्ति गुणकाराः ।
यःक्षेपसोऽपचयोऽपचयः क्षेपश्च विपरीते ॥

क्षेप means *gain*, अपचय means *loss*.

This is an instance of laconic brevity. It means: "Multiplication means division, division becomes multiplication, what is gain becomes loss what is loss, gain." Aryabhaṭṭa has not given any example to illustrate the use of this rule, but we find that in "Līlāvatī" there is a problem that illustrates the use of this method of inversion. It is this:

* यस्त्रिघ्नस्त्रिभिरन्वितः स्वचरणैर्भक्तस्ततः सप्तभिः ।
स्वत्र्यंशेन विवर्जितः स्वगुणितो हीनोद्विपञ्चाशता ॥
तन्मूलेऽष्टयुतेपि दशभिर्जातं द्वयं ब्रूहि तम् ।
राशिं वेत्सि हि चंचलाक्षि विमलां बाले विलोमक्रियाम् ॥

Translation:—Oh girl with beaming eyes, tell me as thou understandest the right method of inversion, which is the number which multiplied by 3, then increased by three-fourth of the product, divided by 7, diminished by 52, the square-root extracted, addition of 8, and division by 10 gives the number 2?

The process consists in calculating with 2 backwards.

For $\frac{\{[\sqrt{(10\times 2-8)^2+52}]\times \frac{3}{2}\times 7\times \frac{4}{7}\}}{3}=28.$

Of course the method of inversion at the time of Bhāskarācārya was more elaborate and contained square and square-root as well.

* This problem has been wrongly given as that of Āryabhaṭṭa by Cajori in his History of Mathematics p. 92. It is that of Bhāskarācārya.

(4)

राश्यूनं राश्यूनं गच्छधनं पिण्डितं पृथक्त्वेन ।
व्येकेन पदेनहृतं सर्वधनं तद्भवत्येव ॥

राशि...a heap, collection.

If x, y, z ....etc. are n heaps and their sums are taken such that of each of them is omitted every time. Sum of all such sums divided by $(n - 1)$ gives the total of all the heaps.

$$x + y + z + \ldots\ldots \text{ to } (n - 1) \text{ letters} = a$$
$$y + z + u + \ldots\ldots \quad ,, \quad ,, \quad ,, \quad = b$$
$$\ldots\ldots\ldots\ldots \quad =$$
$$\ldots\ldots\ldots\ldots \quad =$$

Taking sum of these it would be found that each of the letters occurs $(n - 1)$ times, and therefore the result given is true.

(5)

Here is the deduction from the formula

$$(a + b)^2 = a^2 + b^2 + 2ab$$

संपर्कस्य वर्गाद्विशोधयेदेव वर्गसंपर्कम् ।
यत्तस्य भवत्यर्धं विद्याद्गुणकार संवर्गम् ॥

Translation:—Difference of the square of addition (of quantities) and the addition of (their) squares, divided by two gives the sum of the products (two at a time)

Thus it states

$$\frac{(a + b + c + \ldots\ldots)^2 - (a^2 + b^2 + c^2 + \ldots\ldots)}{2} = \Sigma\, ab.$$

(6)

द्विकृतिगुणात्सवर्गाद्यन्तरवर्गेण संयुतान्मूलम् ।
अन्तरयुक्तं हीनं तद्गुणकारद्वयं दलितम् ॥

कृति = square. द्विकृति $= 2^2 = 4$

Translation :—Square-root of four times the product of the two (quantities) plus the square of (their) difference, added or subtracted by the difference and divided by two, gives the two quantities : i. e. it states the following rule

$$\frac{\sqrt{\{4ab + (a-b)^2\}} \pm (a-b)}{2} = a \text{ or } b.$$

(7)

Here is a formula giving the sum of an Arithmetic Progression and its middle term.

इष्टं व्येकं दलितं सपूर्वमुत्तरगुणं समुखमध्यम् ।
इष्टगुणितमिष्टधनं त्वथवाद्यन्तं पदार्धहतम् ॥
(लब्धधनं अत्र विशेष्यम् ।)

आदि = पूर्व = मुख = the first term a say
चय = उत्तर = common difference d say
पद = गच्छ = इष्ट = the number of terms n say
लब्धधनं = the sum. S_n or simply S say

The first line states: the number of terms minus one, multiplied by the common difference and added to the first term gives the middle term (of an A. P.)

i. e. $$\left[\left(\frac{n-1}{2}\right) d + a\right]$$

is the middle term. The second line gives:

If this middle term is multiplied by the number of terms it is the sum

or

Half the number of terms multiplied by the sum of the first and the last terms is the sum.

$$\text{i. e. } \left[a + \frac{n-1}{2} d\right] n = S_n$$

$$\text{or } \frac{a+l}{2} n = S_n$$

where l is the last term.

The two forms are necessary because the A. P. at times may not have the middle term.

(8)

Here is a formula giving the value of n the number of terms when the sum of the series, the first term and the common difference are given

गच्छोष्टोत्तरगुणिताद्द्विगुणाद्युत्तर विशेषवर्गयुतात् ।
मूलं द्विगुणाद्यूनं स्वोत्तरभजितं स्वरूपार्धम् ॥

Translation :—Square-root of eight times the product of the sum and the common difference together with the square of the difference of double the first term and the common difference, less double the first term, whole divided by the common difference and added one to it, half of this, is the number of terms.

$$\text{i. e. } n = \tfrac{1}{2}\left[\frac{\sqrt{8ds + (2a - d)^2} - 2a}{d} + 1\right]$$

It should be noted down that the very rule has been given in Bakṣāli in a slightly different manner. Brahmagupta's rule is the same as that of Bakṣāli, it is this :

उत्तरहीना द्विगुणादि शेषवर्गं धनोत्तराष्टवधे ।
प्रक्षिप्य पदं शेषोनं द्विगुणोत्तर हृतं गच्छः ॥

$$n = \frac{\sqrt{(2a - d)^2 + 8ds} - (2a - d)}{2d}$$

The above rules are derived by solving a quadratic equation.

(9)

The formulæ for the sum of the squares and the cubes of natural numbers have been given in the following verse.

सैकसगच्छपदानां क्रमात् त्रिसंवर्गितस्य षष्ठोंऽशः ।
वर्गचितिघनस्य भवेच्चितिवर्गो घनचितिघनश्च ॥

In translating note the word सैकसगच्छपदानां क्रमात् the meaning to be taken can be explained thus

पद, सैकपद, सैकसगच्छपद

गच्छ or पद् is the number of terms n

सैकपद $= (n + 1)$

सैकसगच्छपद $= 1 + n + n = (2n + 1)$

$n, (n+1), (2n+1)$] is the order (i. e. क्रम)

The product of these divided by 6 is the sum of the squares of the first n natural numbers.

And the square of the sum of the first natural numbers is the sum of the cubes of the first n natural numbers.

Thus the verse gives

$$1^2 + 2^2 + 3^2 + \dots\dots\dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\text{and } 1^3 + 2^3 + 3^3 + \dots\dots\dots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Compare this with Brahmagupta's rule

द्विगुणपदसैकगुणितं तत् त्रिहृतं भवति वर्गसंकलितम् ।
घनसंकलितं तत्कृतिरेषां समगोलकैश्चितयः ॥

It is quite evident that Brahmagupta's work does not suffer from lack of expression as that of Āryabhaṭṭa.

SECTION II

BRAHMAGUPTA

About hundred years later than Āryabhaṭṭa, the Indian Mathematics reached a still higher degree of perfection. It was Brahmagupta who developed it. Not only did he derive the results obtained by Aryabhaṭṭa and made them elaborate but he gave a quota of his own to the science. The results he obtained stand unique not only in history of Indian Mathematics but also in that of the world Mathematics. Brahmagupta was the first to derive certain results which were troubling the brains of the Western Mathematicians as late as the 17th century. His work is quite lucid and elaborate and does not suffer from the brevity of expression. Only the further developments have been recorded here and not the whole work though it would be observed that the problems he dealt with, show a variety of interest.

(1)

Here is a further extension of the application of the theory of Arithmetic Progression.

एकोत्तरमेकाद्यं यदीष्टगच्छस्य भवति सङ्कलितम् ।
तद्द्वियुत गच्छगुणितं त्रिहृतं सङ्कलितसंकलितम् ॥

If one is the first term and one is the common difference then the sum to n terms is

$$S_n = \frac{n(n+1)}{2}$$

This form is already given but the above rule gives the sum of

$$\sum_{1}^{n} S_n = \frac{n(n+1)}{2} \cdot \frac{n+2}{3}.$$

(2)

But Brahmagupta stands high in esteem for his contribution to Geometry. Of course whenever he dealt with a quadrilateral, he took regular figures such as a square, a rectangle, an isosceles trapezium (i. e. the figures having equal diagonals). These quadrilaterals he designated as अविषमचतुरस्र. These figures are obivously cyclic. The other type of quadrilateral he came across is an irregular but a cyclic quadrilateral. Thus in general Brahmagupta deals with only cyclic quadrilateral. He makes it clear in the 27th stanza of his work on Geometry. Mark the phrase he makes use of, there viz.

चतुर्भुजकोणस्पृशाग्वृत्त

This clearly shows that he worked out problems of cyclic quadrilateral alone. All quadrilaterals are not cyclic, but when he definitely states that his work would deal with only cyclic ones, it is improper on the part of Cajori to say "Brahmagupta gives the area of a quadrilateral which holds good for cyclic one only". Perhaps he might have made such a sweeping assertion by simply depending upon the wrong interpretation by Colebrooke or Kaye.

Thus it would be observed that all his problems on quadrilaterals are based upon this data that they are cyclic. Hence given this condition, only four more conditions are essential to know a quadrilateral.

The area of cyclic quadrilateral is given by the rule

भुजयोगार्धचतुष्टय भूजोनघातात्पदं सूक्ष्मम् ।

Translation :—The square-root of the product of four quantities which are formed by subtracting each side from half the sum of all sides, is the area of a quadrilateral.

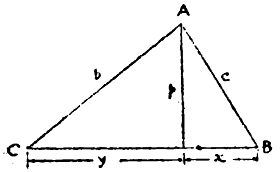
i. e. it is $= \sqrt{\{(s-a)(s-b)(s-c)(s-d)\}}$.

(3)

भुजकृत्यन्तर भूहृतहीनयुता भूर्द्विभाजिताऽबाधे ।
स्वाबाधावर्गोनात् भुजवर्गान्मूलमवलम्बः ॥

युत means addition. हीन means subtraction.

आबाध means segment. अवलम्ब means altitude.



This states that if ABC is a triangle and p, the altitude that divides the base into two parts x and y then

$$y \text{ or } x = \frac{1}{2}\left[a \pm \frac{b^2 - c^2}{a}\right]$$

For, $y = b \cos C = b\left(\frac{a^2 + b^2 - c^2}{2ab}\right)$

$$= \frac{1}{2}\left[a + \frac{b^2 - c^2}{a}\right]$$

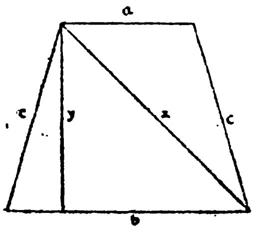
and similarly $x = \frac{1}{2}\left[a + \frac{c^2 - b^2}{a}\right]$

and the second line states $p = \sqrt{b^2 - y^2} = \sqrt{c^2 - x^2}$

(4)

अविषमचतुरस्रभुजप्रति वधयोर्युते पदं कर्णः ।
कर्णकृतिभूमुखयुति दलवर्गोना पदं लम्बः ॥

अविषमचतुरस्र means either a square, a rectangle or an isosceles trapezium i. e. the figure in which both the diagonals are equal.



If x is the diagonal, y is the perpendicular distance between the parallel sides a, b; and c the length of each of the equal sides, the rules given are

$$x = \sqrt{(ab + c^2)}$$

and $$y = \sqrt{x^2 - \left(\frac{a+b}{2}\right)^2}$$

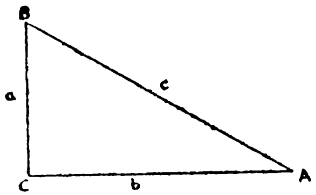
From the property of the rt. ∠ed △

$y^2 = c^2 + \left(\frac{b-a}{2}\right)^2$ and $x^2 - \left(\frac{a+b}{2}\right)^2 = y^2$ these results follow at once.

(5)

The Sulva theorem on a right angled triangle is given as

कर्णकृतेः कोटिकृतिं विशोध्यमूलं भुजोभुजस्यकृतिम् ।
प्रोह्यपदं कोटिः कोटिबाहुकृति युति पदं कर्णः ॥



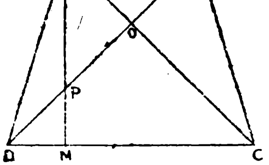
This rule gives

$b = \sqrt{c^2 - a^2}$, $a = \sqrt{c^2 - b^2}$

and $c = \sqrt{a^2 + b^2}$.

(6)

कर्णयुतावूर्ध्वाधरखण्डे कर्णावलम्बयोगे वा ।
स्वाबाधे स्वयुतिहृते द्विधा पृथक्कर्णलम्ब गुणे ॥



If ABCD is the trapezium, in which AB = a, CO = b, AC = BD = x, O is the point of intersection of AC and BD and OA = OB = p, and AM perpendicular to DC cuts BD in P and DP = q.

Then the above rules give

$$x-p=\frac{\left(\frac{b}{2}x\right)}{\left(\frac{a+b}{2}\right)}, \qquad q=\frac{\left(x\,\frac{b-a}{2}\right)}{\left(\frac{a+b}{2}\right)}.$$

These results are based upon the property of similar triangles.

$\triangle$s OAB and ODC are similar

$$\therefore \qquad \frac{OA}{OC}=\frac{AB}{CD}.$$

$$\therefore \qquad \frac{x}{x-p}=\frac{a}{b}.$$

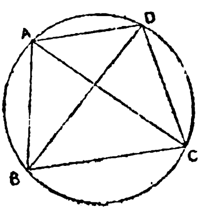
$$\therefore \qquad x-p=\frac{bx}{a+b}$$

Similarly $\triangle$s DPM and ABP are similar whence follows the second result.

(7)

अविषम पार्श्वभुजगुणः कर्णौ द्विगुणावलम्बकाविभक्तः ।
हृदयं विषमस्य................................ ॥

This line gives the value of the radius of the circum-scribing circle of अविषम चतुरस्र.



If x is the diagonal of an isosceles trapezium and y the perpendicular distance between parallel sides then the circumradius of the quadrilateral is given by

$R=\frac{cx}{2y}$ where c is the length of one of the equal sides.

For, if O be the centre

$$\frac{x}{2R} = \text{Sin}\,\frac{A\widehat{O}C}{2}$$

$$= \text{Sin}\,A\widehat{B}C$$

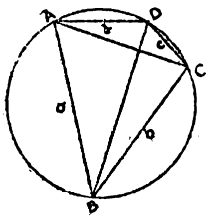
$$= \frac{y}{c}.$$

$$\therefore\quad R = \frac{cx}{2y}.$$

(8)

Here is the most important rule giving the values of the diagonals of any cyclic quadrilateral.

कर्णाश्रितभुजघातैक्यमुभयथाऽन्योऽन्य भाजितं गुणयेत् ।
योगेन भुजप्रतिभुज वधयोः कर्णौ पदे विषमे ॥



If x and y are the diagonals of a cyclic quadrilateral ABCD, whose sides are of length a, b, c, d then the above rule gives

$$x = \sqrt{\frac{ad+bc}{ab+cd}(ac+bd)}$$

$$\text{and}\quad y = \sqrt{\frac{ab+cd}{ad+bc}(ac+bd)}.$$

Hence it would be observed that if we take the product of x and y we get

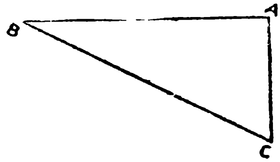
$$xy = ac + bd$$

which is known as Ptolemy's theorem.

(9)

Here are given the sides of any triangle.

इष्टद्वयेन भक्तो द्विधेष्टवर्गः फलेष्टयोगार्धे ।
विषमत्रिभुजस्य भुजाविष्टोनफलार्धयोगोभूः ॥



Let u, v, w be three quantities, then the sides of the triangle ABC are given by

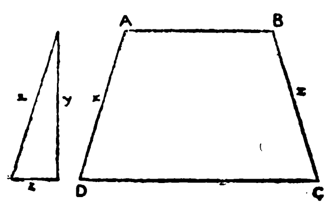
$$\frac{1}{2}\left(\frac{u^2}{v}+v\right),\ \frac{1}{2}\left(\frac{u^2}{w}+w\right) \text{ and}$$

$$\frac{1}{2}\left[\left(\frac{u^2}{v}-v\right)+\left(\frac{u^2}{w}-w\right)\right].$$

(10)

आयतकर्णोबाहू भुजकृतिरिष्टेन भाजितेष्टोना ।
द्विहृता कोट्याधिका भूर्मुखमुना द्विसमचतुरस्रे ॥

The sides of an isosceles trapezium are given in terms of the sides of a right angled triangle.



Let x, y, z be the sides of right angled triangle, x being the hypotenuse. Then the sides of a trapezium ABCD are given by

$$AD = BC = x$$

$$AB = \tfrac{1}{2}\left(\frac{z^2}{a}-a\right)-y$$

$$CD = \tfrac{1}{2}\left(\frac{z^2}{a}-a\right)+y$$

a being any supposed quantity (इष्ट)

(11)

Here is a formula that gives the volume of a cone or a pyramid

क्षेत्रफलं वेधगुणं समखातफलं हृतं त्रिभिः सूच्याः ।

वेध means depth or height. समखात— prism or regularly excavated space. The rule states that the volume of a prism is the product of the area of the base and the height and one-third of that is the volume of a cone or a pyramid having the same base as that of a prism.

While translating this line, Colebrooke has translated the word सूचि as needle.

(12)

Another important addition that Brahmagupta makes, is the accurate value of the volume of the frustum of a pyramid. In giving it he introduces the following terminology

(1) व्यावहारिक फल...practical i. e. approximate
(2) औत्र फल ...(better) approximation
(3) सूक्ष्मफल ...correct or accurate.

Instead of the word व्यावहारिक फल Mahāvir introduces the word कर्मान्तिक फल and for औत्र he uses the word औण्ड्र— this word is not Sanskrit. But taking into consideration that Mahāvir was Jain, it is likely that he made use of the distorted word. The rule is as follows :

मुखतलद्युतिदलगणितं वेधगुणं व्यावहारिकं गणितम् ।
मुखतलगणितैक्यार्धं वेधगुणं स्याद्गणितमौत्रम् ॥
औत्रगणिताद्विशोध्य व्यवहारफलं भजेत् त्रिभिः शेषम् ।
लब्धं व्यवहारफले प्रक्षिप्य भवति फलं सूक्ष्मम् ॥

Square-root of the area of the upper base is called मुख द्युति. Square-root of the area of the base is called तल द्युति.

मुखतल द्युति गणित means the square of the sum of the square root of the base and the upper face.

Thus if Δ and Δ' are the areas of the top and the bottom of a frustum, the व्यावहारिक फल V is given by

$$V = \left(\frac{\sqrt{\Delta} + \sqrt{\Delta'}}{2}\right)^2 h.$$

where h is the height of the frustum.

If A is the औत्रफल

$$A = \frac{\Delta + \Delta'}{2} h,$$

while the सूक्ष्म or the correct value is given by

$$\frac{A - V}{3} + V = \frac{A}{3} + \frac{2V}{3}$$

$$= \frac{h}{6} (\Delta + \Delta') + \frac{h}{6}(\sqrt{\Delta} + \sqrt{\Delta'})^2$$

$$= \frac{h}{3} (\Delta + \Delta' + \sqrt{\Delta\Delta'})$$

SECTION III

MAHĀVIRACARYA

The fame of Mahāvir rests on his brilliant work known as Gaṇitasār Saṅgraha. From the comparison of his work and that of Brahmagupta, it seems that·Mahāvirācārya was familiar with the latter's work; but he has improved upon it a lot. His work is widely known in southern India. He lived about two centuries after Brahmagupta and two centuries before Bhaskarācārya; but it appears that Bhāskarācārya did not know about Mahāvirācārya probably because he was a Jain.

Mahāvir has dealt with almost all the problems his predecessors tackled. He made the classification of arithmetical operations simpler and gave a number of examples to elucidate the rules. His treatment of the work appears quite lucid and elaborate. Almost all the results given by Brahmagupta have been fully dealt with and in addition he gave something more as his quota to the progress of Mathematics in India. Only the additions he made to the science till his time would be dealt with.

It was Mahāvir who first treats the series in Geometric Progressions and gives almost all the formulæ required therein.

The following verse gives the sum of a G. P. to any number of terms.

(1)

गुणसङ्कलितान्त्यधनं विगतैकपदस्य गुणधनं भवति।
तद्गुणगुणं मुखोनं व्येकोत्तर भाजितं सारम् ॥

अन्त्यधन—the value of the last term.

गुण—common ratio.

The verse states that

$$S_n = \frac{ar^{n-1} \times r - a}{r-1}$$
$$= \frac{a(r^n - 1)}{r-1}.$$

where a is the first term and r is the common ratio and S_n is the sum to n terms.

(2)

मुखभक्ते गुणवित्ते यथा निरग्रं तथा गुणेन हृते ।
यावत्योऽत्रशलाका तावन्गाच्छो गुणधनस्य ॥

गुणधन is the last term multiplied by the common ratio i. e. ar^n. This should be divided by the first term and then go on dividing the result by the common ratio till the quotient is one. The number of times this will have to be done gives the number of terms of the series. Because the idea of any big index as not realised in practice and therefore the method followed was rather crude. There is another rule as well that gives the value of n.

एकोनगुणाभ्यस्तं प्रभवहृतं रूपसंयुतं वित्तम् ।
यावत्कृत्वोभक्तं गुणेन तद्वारसम्मितिर्गच्छः ॥

Translation :—Multiply the sum by the common ratio diminished by one, divide this product by the first term, add one. The number of times this is divisible by the common ratio gives the number of terms.

$$\left[\frac{a(r^n - 1)}{r-1}\right](r-1) = a(r^n - 1).$$

$$\text{then} \quad \frac{a(r^n - 1)}{a} = r^n - 1.$$

$$(r^n - 1) + 1 = r^n.$$

Divide r^n by r repeatedly to know the power of r. That is how n was obtained. There are other sūtras too but they are the different forms of one and the same rule.

(3)

Here are some rules to split up a fraction into partial fractions.

वाञ्छाहत युतिहारश्छेदः सव्येक वाञ्छयाप्तोन्यः ।
फलहारहारलब्धे स्वयोगगुणिते हरौऽवास्तः ॥
इष्टगुणांशोऽन्यांशदायुतः शुद्धहृतं फलांशेन ।
इष्टाप्तयुति हरघ्नोहरः परस्यतु तदिष्टहृतिः ॥

The first gives: when $\frac{1}{n}$ is the sum of two intended fractions the fractions are

$$\frac{1}{pn} \text{ and } \frac{1}{\left(\frac{pn}{n-1}\right)}$$

where p is any number : and

$$\frac{1}{mn} = \frac{1}{m(m+n)} + \frac{1}{n(m+n)}$$

while the second verse gives: if $\frac{m}{n}$ is the sum of two intended fractions with a and b as their numerators, then the fractions are

$$\frac{a}{\frac{ap+b}{m} \cdot \frac{n}{p}} \quad \text{and} \quad \frac{b}{\frac{ap+b}{m} \cdot \frac{n}{p} \cdot p}$$

where p is any number so chosen that $(ap+b)$ is divisible by m.

(4)

Mahāvir's work differs from that of others in respect of the definitions he introduced of various figures. He has given the definitions of a triangle—equilateral,

isosceles and scalene—a square, a rectangle, isosceles trapezium, trapezium with three sides equal, a quadrilateral, a circle, a semicircle, an ellipse, a hollow hemisphere and the lune. It is true that the results he derived regarding the area of an ellipse and the length of the curve of an ellipse are not accurate; but as a pioneer in this line his place stands high. Almost all the properties of the cyclic quadrilateral derived by Brahmagupta, have been more lucidly explained by him.

SECTION IV

BHĀSKARĀCĀRYA

Bhāskarācārya is the greatest Hindu Mathematician and it is because of him that India has given a definite quota to forward world march of the science. His exposition of the subject is quite lucid and the obscurity so far found in earlier works has completely disappeared. The logical development and the consecutive arrangement of the subject so far absent from the Hindu works, is found in his work and the most interesting feature of his work is that he combined the cold logic with the richness of poetic imagery. Of course some of the problems he dealt with, have been tackled by his predecessors but he has acknowledged his indebtedness to them like a true scholar.

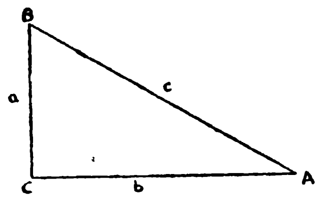
The original works that he did, are the verious solutions of a right angled triangle, the area of the surface of a sphere, the volume of a sphere and sines and cosines of compound angles and the theory of surds. But his place stands high, because of the methods he gave for the solutions of indeterminate problems of the first degree and of the second degree. The rules given are in effect the same as those given by the modern European Algebraist for the solutions of the indeterminate problems of the first degree. It must be borne in mind that Bhāskarācārya gave these methods in the 12th century, while the modern European methods are rediscovered in succession by M. Bachet de Mazeriac, Fermat, Euler and De La Grange in the 17th century. When such is the case why should we grudge to record the credit to the Indian Mathematician that is legitimately his? In the face of all this evidence to maintain that the rules for the solutions of indeterminate

problems might have been borrowed from Greek or Modern European writers, would be absurd.

It is proposed here to enumerate the properties at times without the actual sūtras. The lucidity of exposition have made these Sūtras rather lengthy, but wherever important developments are concerned, the text would be given to guide the reader; otherwise only the methods of working the examples and the translations of the sūtras would be explained.

I

The Right Angled Triangle



(a) If ABC is a right angled triangle the well-known Śulva theorem is given as

$$c = \sqrt{a^2 + b^2},\ a^2 = \sqrt{c^2 - b^2} \text{ and } b = \sqrt{c^2 - a^2}.$$

(b) a being given to find b and c

$$b = \frac{2ax}{x^2 - 1},\ bx - a = c;$$

$$\text{or } b = \frac{\frac{a^2}{x} - x}{2},\ \frac{\frac{a^2}{x} + x}{2} = c.$$

where x is any quantity.

(c) If c is given

$$b = \frac{2cx}{x^2+1}, \; xb - c = a$$

or
$$c - \frac{2c}{x^2+1} = b, \frac{2cx}{x^2+1} = a$$

where x is any quantity.

(d) If x is any quantity such that $x = c \pm b$ then

$$b = \frac{x - \dfrac{a^2}{x}}{2}, \; c = \frac{x + \dfrac{a^2}{x}}{2}$$

or if $x = a + b$, $\dfrac{x - \sqrt{2c^2 - x^2}}{2} = a$, $\dfrac{x + \sqrt{2c^2 - x^2}}{2} = b$.

(e) $a = 2pq$, $b = p^2 - q^2$ and $c = p^2 + q^2$.

These properties have been followed by a number of examples clothed in poetic language.

II

The Value of π

The value of π given by Bhāskarācārya is the same as given by Āryabhaṭṭa I with common factors being cancelled. Thus he takes

$$\pi = \frac{3927}{1250}$$

Then he proceeds to give the area of a circle, the area of the surface of a sphere and the volume of a sphere in one and the same stanza:

वृत्तक्षेत्रे परिधि गुणितव्यासपादः फलं तत् ।
क्षुण्णं वेदैरुपरि परितः कन्दुकस्येव जालम् ॥
गोलस्यैवं तदपिच फलम् पृष्ठजं व्यासनिघ्नम् ।
षड्भिर्भक्तं भवति नियतं गोलगर्भे घनाख्यम् ॥

the first line gives the area of a circle as

$$\text{(circumference)} \times \frac{1}{4}\text{ (diameter)} = 2\pi r \times \frac{2r}{4} = \pi r^2$$

The second line gives the area of the surface of a sphere as

$$= \text{(Area of a great circle)} \times 4$$

$$= 4\pi r^2 \text{ (here Veda means 4)}$$

where r is the radius of the sphere. The last two lines give the volume of a sphere, as

$$\frac{\text{(the surface of the sphere)} \times \text{(diameter)}}{6} = \frac{4\pi r^2 \times 2r}{6}$$

$$= \frac{4}{3}\pi r^3.$$

III

Surds

There are a number of rules dealing with problems on surds. They deal with addition, subtraction, multiplication, division, square and square-root.

(i) To find the value of $\sqrt{a} \pm \sqrt{b}$

Rule :— call $(a+b)$ the greater quantity and if ab is rational call $2\sqrt{ab}$ the smaller one. Then the square-root of the addition of the greater and the smaller quantities is the addition of the two surds ; square-root of the difference of the greater and the smaller quantities is the difference of the two surds. Thus

$$\sqrt{a} + \sqrt{b} = \sqrt{\{(a+b) + 2\sqrt{ab}\}}.$$

and $$\sqrt{a} - \sqrt{b} = \sqrt{\{a+b - 2\sqrt{ab}\}}.$$

Otherwise divide a by b and write down $\sqrt{\frac{a}{b}}$ in two places, add one and subtract one and then the addition and subtraction of the two surds is given by

$$\sqrt{a} + \sqrt{b} = \sqrt{\left\{ \left(\sqrt{\frac{a}{b}} + 1\right)^2 b \right\}}.$$

$$\sqrt{a} - \sqrt{b} = \sqrt{\left\{ \left(\sqrt{\frac{a}{b}} - 1 \right)^2 b \right\}}.$$

Both the rules are in fact one and the same. Whichever was convenient, was made use of.

(ii) For the product of two surds, take the product in the ordinary way and proceed by the methods of addition or subtraction given before.

Ex :—Solve $(\sqrt{2}+\sqrt{3}+\sqrt{8})\ (\sqrt{3}+\sqrt{25})$

One factor is written in the horizontal line and the other in the vertical line and the product is effected as shown below.

	$\sqrt{2}$	$\sqrt{3}$	$\sqrt{8}$
$\sqrt{3}$	$\sqrt{6}$	$\sqrt{9}$	$\sqrt{24}$
$\sqrt{25}$	$\sqrt{50}$	$\sqrt{75}$	$\sqrt{200}$

Product is $3+\sqrt{54}+\sqrt{450}+\sqrt{75}$

For $\sqrt{6}+\sqrt{24}=\sqrt{54}$ and $\sqrt{50}+\sqrt{200}=\sqrt{450}$.

(iii) The method of division and the denominator are the same as the modern one i. e. the numerator and the denominator are to be multiplied by the conjugate of the denominator so that the denominator is rationalised.

Ex :— Solve $\dfrac{\sqrt{9}+\sqrt{54}+\sqrt{450}+\sqrt{75}}{5+\sqrt{3}}$

Multiplying the denominator by its conjugate, we have the expression.

$$= \frac{(\sqrt{9}+\sqrt{54}+\sqrt{450}+\sqrt{75})(5-\sqrt{3})}{(5+\sqrt{3})\ (5-\sqrt{3})}$$

$$= \frac{1}{22}\left\{ \sqrt{225}+\sqrt{1350}+\sqrt{11250}+\sqrt{1875}-\sqrt{27} - \sqrt{162}-\sqrt{1350}-\sqrt{225} \right\}$$

$$= \frac{(\sqrt{11250}-\sqrt{162})+(\sqrt{1875}-\sqrt{27})}{22}$$

Now $1875 + 27 = 1902 = (a + b)$

$1875 \times 27 = 50625 = ab$

$\sqrt{50625} = 225 \quad (= \sqrt{ab}), \quad 225 \times 2 = 450.$

$1902 - 450 = 1452 \quad [= (\sqrt{a} - \sqrt{b})^2].$

$\therefore \quad \sqrt{1875} - \sqrt{27} = \sqrt{1452}.$

Similarly

$\sqrt{11250} - \sqrt{162} = \sqrt{8712}.$

Hence the given fraction

$$= \frac{\sqrt{1452} + \sqrt{8712}}{22} = \frac{\sqrt{18} + \sqrt{3}}{1}$$

$\sqrt{18}$ is further split up into $\sqrt{2} + \sqrt{8}$.

$$\text{Thus } \frac{\sqrt{9} + \sqrt{54} + \sqrt{450} + \sqrt{75}}{5 + \sqrt{3}} = \frac{\sqrt{2} + \sqrt{3} + \sqrt{8}}{1}$$

(iv) The rule for finding the square-root of a binomial surd is put in the form

$$\sqrt{(a + \sqrt{b})} = \left[\sqrt{\frac{a + \sqrt{a^2 - b}}{2}} + \sqrt{\frac{a - \sqrt{a^2 - b}}{2}}\right]$$

To illustrate the rule the example given is this:

Ex:—Find $\sqrt{(5 + \sqrt{24})}$

Here $a = 5, \; b = 24.$

$a^2 = 25. \qquad \therefore \; a^2 - b = 25 - 24 = 1$

$$\frac{a + \sqrt{a^2 + b}}{2} = 3$$

$$\frac{a - \sqrt{a^2 - b}}{2} = 2$$

Hence $\sqrt{(5+\sqrt{24})}=(\sqrt{3}+\sqrt{2})$

If it is an example of the type

$$\sqrt{\{a+\sqrt{b}+\sqrt{c}+\sqrt{d}\}}$$

then the method followed was to consider

$$a^2-(b+a);\ a^2-(c+d) \text{ or } a^2-(b+d).$$

if one of these expression is a complete square. Thus if

$$a^2-(b+c)=x^2 \text{ say}$$

then $\sqrt{\dfrac{a-x}{2.}}$, $\sqrt{\dfrac{a+x}{2}}$ are to be considered.

One of them is to be kept and the other is to be combined with $\sqrt{d}$. The method would be better understood from the example worked out by the author.

Ex:—Find $\sqrt{\{10+\sqrt{24}+\sqrt{40}+\sqrt{60}\}}$.

$10^2=100,\ 100-(24+40)=36,\ \sqrt{36}=6\ (=x)$

$10+6=16,\ 10-6=4,$

$$\frac{16}{2}=8,\ \frac{4}{2}=2.$$

Consider $\sqrt{8}$ and $\sqrt{2}$ now. One of these two numbers would be one term of the square-root and the other is to be combined with the term that is left. Leave $\sqrt{2}$ as it is and proceed with 8.

$$8^2=64,\ 64-60=4,\ \sqrt{4}=2$$

$$8+2=10,\quad 8-2=6.$$

$$\frac{10}{2}=5,\quad \frac{6}{2}=3$$

$\therefore$ $\sqrt{5}$ and $\sqrt{3}$ are the required results.

$\therefore$ $\sqrt{5}+\sqrt{3}+\sqrt{2}=\sqrt{\{10+\sqrt{24}+\sqrt{40}+\sqrt{60}\}}$

Ex :— Solve

$\sqrt{\{16 + \sqrt{24} + \sqrt{40} + \sqrt{48} + \sqrt{60} + \sqrt{72} + \sqrt{120}\}}$.

The working is as follows :—

$16^2 = 256,\ 256 - (24 + 40 + 48) = 144, \sqrt{144} = 12.$

$16 + 12 = 28, \qquad 16 - 12 = 4.$

$\frac{28}{2} = 14, \qquad \frac{4}{2} = 2$

We have then $\sqrt{14}$ and $\sqrt{2}$. Keep $\sqrt{2}$ as it is and proceed with the number 14.

$14^2 = 196, \qquad 196 - (120 + 72) = 4, \quad \sqrt{4} = 2$

$14 + 2 = 16, \quad 14 - 2 = 12.$

$\frac{16}{2} = 8, \qquad \frac{12}{2} = 6$

We have then $\sqrt{8}$ and $\sqrt{6}$. Keep $\sqrt{6}$ as it is and proceed with the number 8.

$8^2 = 64, \qquad 64 - 60 = 4, \qquad \sqrt{4} = 2$

$8 + 2 = 10, \quad 8 - 2 = 6$

$\frac{10}{2} = 5 \qquad \frac{6}{2} = 3.$

We have $\sqrt{5}$ and $\sqrt{3}$ and all the terms are exhausted.

$$\therefore \ (\sqrt{2} + \sqrt{6} + \sqrt{5} + \sqrt{3}) = \sqrt{\{16 + \sqrt{24} + \sqrt{40} + \sqrt{48} + \sqrt{60} + \sqrt{72} + \sqrt{120}\}}$$

A number of examples have been worked out by the author by this very method. In order to help the working in general, the author has given the number of terms in the square-root of a surd the number of terms in the square and the number of rational terms in the square. This helps to find how many terms would be there in the square-root.

IV

The Indeterminate Equations

The study of indeterminate equations began from the days of Āryabhaṭṭa, as far as the developments in this period are concerned. Āryabhaṭṭa onwards almost all the Mathematicians have dealt with the subject but the exposition of the methods by Bhāskarācārya is quite lucid. He has treated a variety of problems on it and some of his methods are quite original. For this reason the place of Bhāskarācārya stands high in the galaxy of Mathematicians.

Here are the rules which are translated with explanations wherever necessary.

भाज्योहारः क्षेपकश्चापवर्त्यः केनाप्यादौ संभवे कुट्टकार्थं ।
येनच्छिन्नौ भाज्यहारोन तेन क्षेपश्चैतद्दुष्टमुद्दिष्टमेव ॥

परस्परं भाजितयोर्ययोर्यः शेषस्तयोः स्यादपवर्तनं सः ।
तेनापवर्तेन विभाजितौ यौ तौ भाज्यहारौ दृढसंज्ञकौस्तः ॥
मिथो भजेत्तौ दृढभाज्यहारौ यावद्विभाज्ये भवतीहरूपम् ।
फलान्यधोऽधस्तदधो निवेश्य क्षेपस्तथान्ते खमुपान्तिमेन ॥
स्वोर्ध्वेहतेऽन्त्येन युते तदन्त्यं त्यजेन्मुहुः स्यादिति राशियुग्मं ।
उर्ध्वो विभाज्येन दृढेन तुष्टः फलं गुणस्यादधरो हरेण ॥
एवं तदेवात्र यदा समस्ताः स्युर्लब्धयश्चेद्विषमास्तदानीम् ।
यथागतौ लब्धिगुणौ विशोध्यौ स्वतक्षणाच्छेषमितौ तु तौस्तः ॥

भवति कुट्टविधेर्युति भाज्ययोः समपवार्तितयोरथ वा गुणः ।
भवति यो युतिभाजकयोः पुनः सच भवेदपवर्तनसंगुणः ॥
क्षेपजे तक्षणाच्छुद्धे गुणाप्तीस्तौ विशुद्धिजे ।
गुणलब्ध्योः समं ग्राह्यं धीमते लक्षणे फलम् ॥
हरतष्टे घनक्षेपे गुणलब्धी तु पूर्ववत् ।
क्षेपतक्षण लाभाढ्या लब्धी शुद्धौ तु वर्जिता ॥
अथवा भागहारेण तष्टयो क्षेपभाज्ययोः ।
गुणः प्राग्वत्ततो लब्धी भाज्यादृधृत युतोद्धृतात् ॥
क्षेपे तु रूपे यदि वा विशुद्धौ स्यातां क्रमाद्ये गुणकार लब्धी ।
अभिप्सिते क्षेप विशुद्धि निघ्ने स्वहारतष्टे भवतस्तयोस्ते ॥

Translation & Explanation:—To find an unknown such that when it is multiplied by a known number and another known number added to it, and if the result is divided by a third known number, remainder is zero (i. e. the division is complete). This type of problem is known as "Kuṭṭaka".

The following terminology is introduced :—

The first known number (i. e. that which multiplies the unknown is known as Guṇaka (गुणक) i. e. multiplier, the other known number that is added to the product is called Kṣepak (क्षेपक) i. e. augment (augment may be positive or negative), the third known number that divides this result is known as Bhājaka (भाजक) i. e. divisor, the result obtained is known as Bhāgākāra (भागाकार). Thus if

$$\frac{ax + c}{b} = y.$$

where x and y are unknowns and a, b, c are known quantities, then

a is known as the multiplier (गुणक)
c is known as the augment (क्षेपक)
b is the divisor (भाजक)
y is the "Bhāgākara (भागाकार) no corresponding terminology for this in English as the word quotient fails to connote the idea involved therein.

First of all remove the common factor or factors of a, b, c and if a', b', c' are the values that remain after this removal, take the equation

$$\frac{a'x + c'}{b'} = y$$

which we can write

$$\frac{ax + c}{b} = y$$

as there is no loss of generality if we supress the dashes. Now divide the multiplier a by b and divide b by the

8

remainder until finally the remainder is one. Write all the quotients derived by this process in a vertical line one below the other and write at the bottom of this the augment and below it zero. This vertical line is known as *valli* (वल्ली) i.e. a creeper. Multiply by the number above zero in this *valli* by a number immediately above it and to the product add zero. Multiply the number thus obtained by the next upper number in the line and to the product add the number below it and so on until all the numbers in the *valli* are exhausted. Of the last two numbers thus obtained, if the lower is applicable to the problem given then the number above is the value of Bhāgākāra i. e. y.

To derive the least values of x and y divide the value of y found above by a the "guṇaka", and call the remainder y' divide the value of x by b and call the remainder x'. Multiply a by x' and to the product add c. Dividing the same by b and the quotient will be the new value of y. To the first remainder add a continuously and to the second remainder b and as many times we shall have new values of x and y.

If the number of quotients that are to be written in the vertical line (i. e. *valli*,) is even, then the above method is applicable: but when it is odd adopt the following process. Take all the steps mentioned above and then subtract the value of y from the multiplier a and that of x from the divisor b, then x will be found right while the value of y would be wrong. To find the correct value of y multiply its wrong value by the common factor of a and c. If b and c have a common factor the value of x first derived should be multiplied by this common factor and the correct value of x will be obtained if the original value is wrong. When the augment c is negative subtract the value of x from the divisor b and that of y from the multiplier a.

If the subtraction is possible, then the question is solved, if it is not, take the excess of the subtrahend above the minuend to be negative, multiply the minuend by a number so that the product may be greater than the negative quantity; from this quantity subtract the negative quantity and the remainder will be the required number.

If a is negative, subtract the value of x from b and that of y from a.

If c is positive and greater than b, subtract b or its multiples from c till a number less than b remains. Add this number of the multiple of b to the value of y and the sum will be its correct value. If $c < 0$ subtract the number of the multiple from the value of y.

If $a > b$ and $c > b$, subtract b or its multiples from both, take the remainders as the new values for a and c and proceed. That would give the correct value of x but a wrong value for y. If in this process, by dividing c by b there is no remainder x will be zero, and y would be $\frac{c}{b}$. If the numbers are not reduced but the quotients are taken from the original numbers, x and y will always be found correct. If they are reduced, x and y will be found correctly when two are reduced, but one of them will be brought out right when both are not reduced.

Of course the rules are very lengthy but they have been given with elaborate exposition to illucidate the calculations. They will better be understood from the examples worked out by the author by making use of these rules.

Example:—Solve $\frac{221x + 65}{195} = y$.

$a = 221, \quad b = 195, \quad c = 65.$

It would be observed that 13 is a common factor between a, b, c. Removing that, their new values are 17, 15 and 5.

Take 17 as the multiplier, 15 the divisor and 5 the augment

Then

$$\begin{array}{r|r|l} 15 & 17 & 1 \\ & 15 & \\ \hline 2 & 15 & 7 \\ & 14 & \\ \hline & 1 & \end{array}$$

The quotients are 1 and 7.

Writing in the *valli* form 1, 7 the augment 5 and 0 below as directed, we have

$$\begin{array}{r|l} 1 & 40 \\ 7 & 35 \\ 5 & \\ 0 & \end{array}$$

For $5 \times 7 = 35, \quad 35 \times 1 + 5 = 40.$
Thus 40 and 35 are the numbers to be dealt with. From 40 subtract the multiple of 17 so that the result is less than 17.

$$\therefore \quad 40 - 2 \times 17 = 6 = y$$

and from 35 subtract the multiple of 15 so that the result is less than 15.

$$35 - 2 \times 15 = 5 = x$$

Hence $x = 5, \quad y = 6.$

For it would be observed that

$$\frac{221 \times 5 + 65}{195} = 6,$$

To find another set of values of x and y

$$17 + 6 = y$$
$$15 + 5 = x$$
$$2 \times 17 + 6 = 40 = y$$
$$2 \times 15 + 5 = 35 = x$$
$$3 \times 17 + 6 = 57 = y$$
$$3 \times 15 + 5 = 50 = x$$

and so on. Thus infinite sets of values for (x, y) can be derived in this manner.

2nd Method:—$a = 221$, $b = 195$, $c = 65$.

Removing the common factor, we have

$$a' = 17, \quad b' = 15, \quad c' = 5.$$

Instead of 5 take $c = 1$

Then

$$\begin{array}{r|l|l} 15 & 17 & 1 \\ & 15 & \\ \hline & 2 & \begin{array}{|l|l} 15 & 7 \\ 14 & \\ \hline 1 & \end{array} \end{array}$$

Writing the quotients in the *valli* form together with the new augment 1 and zero, we have

$$\begin{array}{r|l} 1 & 8 \\ 7 & 7 \\ 1 & \\ 0 & \end{array}$$

Multiply 8 and 7 by 5 the value of c' and we have 40 and 35. Then proceed as before.

If c is negative subtract 7 from 15 and 8 from 17, and 8 and 9 remain. Multiplying these by 5 we get 40 and 45. Then

$$40 - 2 \times 15 = 10 = x$$
$$40 - 2 \times 17 = 11 = y$$

Ex :— Solve $\frac{100x+90}{63}=y$.

63	100 63	1
37	63 37	1
26	37 26	1
11	26 22	2
4	11 8	2
3	4 3	1
1		

Writing the quotients, the augment and zero in the *valli* form we have

1	2430	$1 \times 90 + 0 = 90$
1	1530	$2 \times 90 + 90 = 270$
1	900	$2 \times 270 + 90 = 630$
2	630	$1 \times 630 + 270 = 900$
2	270	$1 \times 900 + 630 = 1530$
1	90	$1 \times 1530 + 900 = 2430$.
90		
0		

The last two numbers are 2430 and 1530. Divide the former by 100 and the latter by 63. The remainders are 30 and 18 respectively

$$\therefore \quad x = 18 \text{ and } y = 30.$$

2nd Method. Divide 100 by 90 and 90 by 10 and then the new values of a, b, c are to be taken

$$a' = 10,\ b' = 63,\ c' = 9$$

63	10	0		
	0			
	10	63	6	
		60		
		3	10	3
			9	
			1	

Putting these in the *valli* form, with the augment and zero we have

0	27		
6	171	i. e.	$9 \times 3 + 0 = 27$
3	27		$27 \times 6 + 9 = 171$
9			$171 \times 0 + 27 = 27$
0			

The last two numbers are 27 and 171.

$$27 - 2 \times 10 = 7$$
$$171 - 2 \times 63 = 45,$$

The number of quotients in the *valli* is odd. Hence subtract 45 from 63 and we get 18 which is the value of x. 7 subtracted from 10 gives 3 which is the wrong value of y. So to find the correct value of y multiply 3 by the common factor 10 and we have $y = 30$.

3rd Method. Common factor between b and c is 9. Removing that, the new values are

$$a = 100, \quad b' = 7, \quad c' = 10.$$

$$\begin{array}{r|c|c|c}
7 & \begin{array}{c}100\\98\end{array} & 14 & \\
\hline
 & 2 & \begin{array}{c}7\\6\end{array} & 3 \\
\hline
 & & 1 &
\end{array}$$

Arranging these quotients in the *valli* form with the augment and zero, we have

$$\begin{array}{r|l}
14 & 430 \\
3 & 30 \\
10 & \\
0 &
\end{array}$$

From 430 subtract 4×100 and we get 30. This is the value of y subtract 4×7 from 30 and 2 remains. But 2 is not the correct value of x. Multiply 2 by the common factor 9 and we have $x = 18$.

4th Method. Removing the common factor from a and c.

$$a' = 10, \quad b' = 63, \quad c' = 9.$$

Removing the common factor from b and c, we get

$$a = 100, \quad b' = 7, \quad c' = 10.$$

Thus uniting the two processes we have

$a' = 10$, $b' = 7$, and c should be taken equal to the difference of the two operations, i. e. $10 - 9 = 1$.

$$a' = 10, \quad b' = 7, \quad c' = 1.$$

Then

$$\begin{array}{r|c|c|c}
7 & \begin{array}{c}10\\7\end{array} & 1 & \\
\hline
 & 3 & \begin{array}{c}7\\6\end{array} & 2 \\
\hline
 & & 1 &
\end{array}$$

writing the quotients in the *valli* form, together with the augment and zero

1	3	$1 \times 2 + 0 = 2$
2	2	$2 \times 1 + 1 = 3$
1		
0		

3 and 2 are the values obtained for x and y respectively. Multiply 2 by 9 the common factor of b and c and we have $x=18$. Multiply3 by 10 the common factor of a and c and we have $y=30$.

V

Simultaneous Indeterminate Equations (मिश्रकुट्टक.)

एको हरश्चेद्गुणकौ विभिन्नौ तदागुणैक्यं परिकल्पभाज्यम् ।
अग्रैकमग्र कृतउक्तवद्यः संश्लिष्टसंज्ञा स्फुटकुट्टकौऽसौ ॥

Translation :—Add the multipliers together and take the sum as the new multiplier, add the two augments and take the sum as the new augment with a negative sign. Keeping the divisor as it is, follow the process already given.

Solve $\frac{a_1 x}{b} = y + c_1 \qquad \frac{a_2 x}{b} = z + c_2$

where $a_1 = 5, \quad c_1 = 7, \; a_2 = 10, \quad c_2 = 14, \quad b = 63$

$a = a_1 + a_2 = 10 + 5 = 15. \; c = c_1 + c_2 = 7 + 14 = 21$

$\therefore \; a = 15, \quad b = 63, \text{ and } c = 21.$

Proceeding as before

$$
\begin{array}{r|r|l}
63 & 15 & 0 \\
 & 0 & \\
\hline
 & 15 & 63 \;|\; 4 \\
 & & 60 \\
\hline
 & & 3 \;|\; 15 \;|\; 5 \\
 & & \quad\;\; 15 \\
\hline
 & & \quad\;\; 0
\end{array}
$$

Writing the first two quotients in the *valli* form together with the augment and zero, we have

0	7
4	28
7	
0	

Hence we get 7 and 28.

$7 - 5 = 2, \quad 5 - 2 = y$ which is wrong.

$21 - 7 = 14 = x \quad x$ is correct.

For $14 \times 5 = 70$ and $\frac{70}{63}$ leaves the remainder 7.

Also $14 \times 10 = 140$

and $\frac{140}{63}$ leaves the remainder 14.

The theory of these equations is so elaborately developed as it was found that it has a bearing upon the problems in Astronomy. For instance there is a theoretical problem which runs thus :—

"A body completes 37 revolutions in 49 days; how many revolutions would it make in 17 days?"

The answer is easily obtained by the rule of three. It is 12 revolutions, 10 signs 1°, $13'$ $28''\frac{8}{49}$ (where 1 sign = 30°).

Suppose that a student is asked to memorise the result; and at a later date the example is lost and the whole answer he forgets except the last fraction $\frac{8}{49}$. Then

the whole answer might be replaced by continuously solving indeterminate equations. Thus first solve

$$\frac{60x - 8}{49} = y$$

from which we get $x = 23$, $y = 28$.

Here a is taken as 60 since $60'' = 1'$. Again solve

$$\frac{60x' - 23}{49} = y'$$

and we find $x' = 11$, $y' = 13$

Then solve $$\frac{60x'' - 11}{49} = y''$$

proceeding in this manner the whole answer can be replaced.

VI

Quadratic Indeterminate Equations

To solve $Ax^2 + B = y^2$ where A and B are knowns.

A is the multiplier

B is the augment

x is the least root ह्रस्व

y is the greatest root ज्येष्ठ.

ह्रस्वज्येष्ठ क्षेपकान्यस्य तेषां तानन्यान्वाऽधो निवेश्य क्रमेण ।
साध्यान्येभ्यो भावनाभिर्बहूनि मूलान्येषां भावनाप्रोच्यते तः ॥
वज्राभ्यासो ज्येष्ठलब्धोस्तदैक्यं ह्रस्वं लब्धोराहतिश्च प्रकृत्या ।
क्षुण्णाज्येष्ठाभ्यासयुग्ज्येष्ठमूलम् तत्राभ्यासः क्षेपयोः क्षेपकस्यात् ॥
ह्रस्वं वज्राभ्यासयोरन्तरं वा लब्धोर्घातोयः प्रकृत्या विनिघ्नः ।
घातोयश्च ज्येष्ठयोस्तद्वियोगो ज्येष्ठं क्षेपोऽत्रापि च क्षेपघातः ॥
इष्टवर्गहृतः क्षेपः क्षेपः स्यादिष्ट भाजिते ।
मूले ते स्तोथवा क्षेपः क्षुण्णः क्षुण्णे तदापठे ॥

इष्टवर्गप्रकृत्योर्यद्विवरं तेन वा भजेत् ।
द्विघ्नमिष्टं कनिष्ठं तत्पदं स्यादेकसंयुतौ ॥
ततो ज्येष्ठमिहानन्त्यं भावनाभिस्तथेष्टतः ।

Rules :—Take any number and call it the "least" root, and square it. Multiply it by the any known number called the multiplier and add or subtract the augment to make it a perfect square. Take its square-root and call it the "greatest" root. Write down these two roots and the augment in one line and below it again the same roots and the augment so that the least roots, the greatest roots and the augments are in the verticular columns respectively. Multiply crossways i. e. the least by the greatest and the greatest by the least roots and add the two products. Call this sum the least root. Take the product of the two least roots and multiply it by the multiplier and add to it the product of the greatest root. The result will be the greatest root; and the product of the two augments will be the new augment.

To obtain another set of values repeat the same process and perform the same operations with the new least and the greatest roots.

2nd Method :—In the above operation after multiplying crossways take the difference and it would be the least root and the difference between the product of the two least roots multiplied by the multiplier and the product of the greatest roots will be the corresponding greatest root.

If the augment of the operation is the same as the original one so far so good; but if it is greater, divide it by the square of an assumed number so that the original augment may be obtained, but if it be less, multiply it by a square of an assumed number so that the original augment

may be obtained. In the first case divide the greatest and the least roots by that assumed number and in the second case multiply them so that they may correspond.

3rd Method:—Assume a number and divide its double by the difference of the multiplier and the square of the number and the quotient will be the least root when unity is the augment; thence obtain the greatest root.

Ex:— Solve $8x^2 + 1 = y^2$

Suppose 1 is the least root. Then

$8 + 1 = 9 \quad \therefore \quad y = 3$

$\therefore$ 3 is the greatest root.

$\therefore$ Writing these, together with augment in horizontal lines as directed by the rule

Least root	Greatest root	Augment
1	3	1
1	3	1

cross multiply as indicated by the arrows and add. Thus

$$3 + 3 = 6 = x$$

the corresponding value of y is 17. For,
the product of the least roots is $1 \times 1 = 1$

$\therefore \quad 1 \times 8 = 8$

adding this to the product of the greatest roots, that is $(3 \times 3 = 9)$, $8 + 9 = 17 = y$ corresponding to $x = 6$.

Again taking the two sets of values of x and y together with the augments according to the rule, we have

Least	Greatest	Augment
1	3	1
6	17	1

Cross multiplying and adding we have

$$17 + 18 = 35 = x$$

The product of the least roots is $1 \times 6 = 6$

$$6 \times 8 = 48$$

The product of the greatest roots $3 \times 17 = 51$

$51 + 48 = 99 = y$ corresponding to $x = 35$

Proceeding further

Least root	Greatest root	Augment
1	3	1
35	99	1

Cross multiplying and adding we have

$$99 + 105 = 204 = x$$

Product of the least roots is $35 \times 1 = 35$

$$35 \times 8 = 280$$

Adding to 280 the product of the greatest roots

i. e. $3 \times 99 = 297$, we have

$$280 + 297 = 577 = y$$

Thus the values for x and y are (1, 3), (6, 17), (35, 99), (204, 577) and so on. We can proceed like this indefinitely and obtain infinite sets of values of x and y.

To illustrate the use of the other rules the example given is

$$11x^2 + 1 = y^2$$

Here if we take $x = 1$ the left hand side is not a perfect square. But if the augment is taken (-2) then the left hand side is a perfect square since $11 - 2 = 9$ and $\sqrt{9} = 3$.

By taking 1 as the least root and 3 as the greatest root and (-2) as the new augment we have

Least root	Greatest root	Augment
1	3	−2
1	3	−2

Cross multiplying and adding we get

$$3 + 3 = 6 = x_1$$

Product of the least roots $1 \times 1 = 1$.

Multiply this 1, by 11 and add to it the product of the greatest roots i. e. $3 \times 3 = 9$

$$\therefore\ 11 + 9 = 20 = y_1$$

But these values x_1 and y_1 satisfy the equation

$$11x_1^2 + 4 = y_1^2$$

The square of an assumed augment $(-2) \times (-2) = 4$. Dividing 4 by 4 we have 1 the original augment. Thus to obtain the values of x and y divide x_1 and y_1 by 2 and we get the values of x and y

$$x = \frac{x_1}{2} = \frac{6}{2} = 3$$

$$y = \frac{y_1}{2} = \frac{20}{2} = 10$$

Otherwise: take 1 the least root and 5 the augment, then the greatest root is 4. Hence we have

Least root	Greatest root	Augment
1	4	5
1	4	5

Cross multiplying and adding we have

$$4 + 4 = 8 = x_1$$

The product of the least roots is $1 \times 1 = 1$. Multiply this 1 by 11 and add to it the product of the greatest roots i. e. $4 \times 4 = 16$ and we have

$$11 + 16 = 27 = y_1$$

where x_1, y_1 satisfy the equation

$$11x^2_1 + 25 = y^2_1$$

Thus the new augment is $5 \times 5 = 25$, and since $\sqrt{25} = 5$; divide this 5 by the augment 5 and we have the original augment 1. Hence the values of x and y are given by

$$x = \frac{x_1}{5} = \frac{8}{5}, \quad y = \frac{y_1}{5} = \frac{27}{5}.$$

Thus the sets of values obtained by the two methods, of the equation

$$11x^2 + 1 = y^2 \text{ are } (3, 10) \text{ and } \left(\frac{8}{5}, \frac{27}{5}\right)$$

Proceeding further we have

Least root	Greatest root	Augment
3	10	1
$\frac{8}{5}$	$\frac{27}{5}$	1

Cross multiplying and adding we have

$$\frac{81}{5} + \frac{80}{5} = \frac{161}{5} = x.$$

Now the product of the least roots is $\frac{8}{5} \times 3 = \frac{24}{5}$

Multiplying this $\frac{24}{5}$ by 11 and adding to it the product of the greatest roots i. e. $10 \times \frac{27}{5} = \frac{270}{5}$ we have

$$11 \times \frac{24}{5} + \frac{270}{5} = \frac{534}{5} = y.$$

Thus another set of values of x and y is $\left(\frac{161}{5}, \frac{534}{5}\right)$

So we can proceed further indefinitely.

Again, after multiplying crosswise and taking the difference we have

$$\frac{81}{5} - \frac{80}{5} = \frac{1}{5} = x$$

The product of the least roots is $\frac{24}{5}$. This multiplied by 11 is $\frac{264}{5}$. The product of the greatest roots is $\frac{270}{5}$.

Then $$\frac{270}{5} - \frac{264}{5} = \frac{6}{5} = y.$$

Thus another set of values for x and y are obtained by the other method.

VII

The Cyclic Method or the Cakravāl Method

ह्रस्व ज्येष्ठ पदक्षेपान्भाज्य प्रक्षेपभाजकान् ।
कृत्वा कल्प्यो गुणस्तत्र तथा प्रकृतितश्च्युते ॥
गुणवर्गे प्रकृत्योनेऽथवाल्पं शेषकं यथा ।
तत्तु क्षेपहृतं क्षेपः व्यस्तः प्रकृतितश्च्युते ॥
गुणलब्धिः पदं ह्रस्वं ततो ज्येष्ठमतोऽसकृत् ।
त्यक्त्वा पूर्वपदक्षेपांश्चक्रवालमिदं जगुः ॥

Assume the least and the greatest roots and the augment and then take the least root as the dividend and the augment the divisor and greatest root the augment, bring out the multiplicand and the "bhāgākār" i. e. the quotient (as in kuṭṭaka). The square of this multiplicand should be subtracted from the given multiplier or this multiplicand should be subtracted from the given multiplier so that the difference may be small. This difference divided by the original augment is to be taken as the new augment, which is reversed if the subtraction be from the original multiplier. The quotient corresponding to the multiplicand will be the least root, whence the greatest root will be deduced; and so on. The operation is repeated by setting aside the former roots and the augment.

Method would be understood by the following example worked by the author.

Example. What is the square which being multiplied by 67 and one being added to it will yield a square-root ?

i. e. solve $67x^2 + 1 = y^2$.

Let 1 be the least root. Then when it is multiplied by 67 and 3 subtracted from it gives us the complete square. Hence with this arrangement 8 is the greatest root.

Then the dividend is 1
the divisor is 3
the augment is 8.

Subtract two times 3 the divisor, from 8 the augment and 2 remain. Take 2 as the augment. The quotient when 1 is divided by 3 and 8 is divided by 3 are 0 and 2 respectively. Subtract the divisor and the dividend from the multiplier of 3 i. e. 2 and we have 1 and 1. As the divisor 3 is subtracted two times from the augment 8 add 2 to the quotient and we get 3; and the multiplier is 1. As 67 cannot go from the square of the multiplier, subtract the square of this from 67 and 66 remain, but this is a big number, so add the divisor 3 twice to the multiplier and we get $1+6=7$. Subtract the square of 7 from 67 and we have $(67 - 49) = 18$. Add the dividend to the quotient and the result is 5. Divide by the augment of operation of multiplication of the square which when reversed is (-3) i. e. $\frac{18}{-3} = -6$ is the quotient. And as the square of 7 has been subtracted from the multiplier 67 the negative sign is to be made positive according to the rule. Thus $+6$ is the augment and 5 is the least root and since

$$67 \times 25 + 6 = 1681 = 41^2$$

41 is the greatest root.

$\therefore$ now take the dividend 5
the divisor 6
and the augment 41

Then perform the operation as before as in the case of the linear indeterminate equation

$$\frac{5\xi + 41}{6} = \eta$$

and we get $\xi = 5$ and $\eta = 11$.

Subtract the square of 5 from 67 and that gives $(67 - 25) = 42$. 42 divided by the augment 6 gives 7 as the quotient which is to be reversed in sign since from the multiplier 67 we have subtracted the square of 7. Then 11 is the value of the least root and the greatest root is given by

$$67 \times 11^2 - 7 = 8100 = 90^2$$

$\therefore$ the greatest root is 90.

Then as before

take the dividend 11
the divisor 7
and the augment 90.

Perform the operation as before

$$\frac{11\xi_1 + 90}{7} = \eta_1$$

and we get $\xi_1 = 2$ and $\eta_1 = 16$.

Subtract the square of 2 from the multiplier 67 and that gives $(67 - 4) = 63$ which is a big number and therefore add the divisor 7 to 2 and we get 9. Subtract the multiplier 67 from the square of 9 and we have $(81 - 67) = 14$. Dividing 14 by the negative augment 7 we have $\frac{14}{-7} = -2$. The new augment is therefore -2.

Adding the value of $\eta_1 = 16$ to 11 we get 27 which is the least root and hence the greatest root is given by

$$67 \times 27^2 - 2 = 48841 = 221^2$$

$\therefore$ 221 is the greatest root.

Hence as before, take the dividend 27

the divisor 2

and the augment 221.

At this stage we can save our calculations by falling back upon the old method.

Arranging these as follows

Least root	Greatest root	Augment
27	221	−2
27	221	−2

Cross multiplying and adding

$221 \times 27 + 221 \times 27 = 11934$ is the least root.

The product of the least roots $27 \times 27 = 729$

$729 \times 67 = 48843$.

Adding to this the product of the greatest roots

i. e. $221 \times 221 = 48841$

we get $48843 + 48841 = 97684$.

The product of the augments is 4. And when this 4 is divided by the square of the augment (-2) we get 1

which is the original augment 1 as given in the equation

$$67x^2 + 1 = y^2$$

Hence dividing 11934 by 2 and also 97684 by 2 we get

$$x = 5967 \text{ and } y = 48842.$$

The analysis of the "cākraval" or the cyclic method can therefore be given as follows:

To solve $Ax^2 + B = y^2$ where A and B are given.

Find α, β and B_1 so that

$$A\alpha^2 + B_1 = \beta^2$$

and then solve the indeterminate equation

$$\frac{\alpha\xi + \beta}{B_1} = \eta$$

and from the known values of α, β and B_1 find ξ and η. If $\xi^2 > A$ take $(\xi^2 - A)$ if not take $(A - \xi^2)$. If this difference is a small number, so far so good; if not take the multiple of B_1 and add it to the value of ξ for a new value till we have

$$[(mB_1 - \xi)^2 - A] \text{ or } [A - (mB_1 - \xi)^2]$$

is a small number. Divide this by B and if the square has been subtracted, change the sign of the quotient. Then take

$$\frac{\xi^2 \sim A}{B_1} \text{ or } \frac{(mB_1 + \xi)^2 \sim A}{B_1} = B_2$$

then the solution of the equation

$$AX^2 + B_2 = Y^2$$

is $X = (m\alpha + \eta)$ with the corresponding value for y.

If $B_2 \neq B$ $B_2 \neq Bp^2$ or $B_2 \neq \frac{B}{p^2}$ proceed as before.

Otherwise follow the old method given, if it is applicable.

Besides the methods given above there are other methods which are applicable only in some special cases.

Rule :—If the multiplier is the sum of two squares, take the square-root of these two squares and by these two numbers divide the augment separately and the results will be the least roots.

Example. Solve $13x^2 - 1 = y^2$

$$\text{Since } 13 = 2^2 + 3^2$$

$$\therefore \quad x = \tfrac{1}{3} \text{ or } \tfrac{1}{2}$$

The corresponding values for y are $\frac{2}{3}$ or $\frac{3}{2}$.

Rule :—If the multiplier is a square, divide the augment by the assumed number and write the quotient in two places and in one place add to it and in the other subtract from it the assumed number and the second is divided by the square-root of the multiplier; halve them both and they will be the greatest and the least roots.

To put it analytically

$$Ax^2 + B = y^2$$

where $A = n^2$

Take r any number then

$$\frac{\frac{B}{r} + r}{2} = y \text{ and } x = \frac{\frac{B}{r} - r}{2n}:$$

For

$$A\left(\frac{\frac{B}{r} - r}{2n}\right)^2 + B = \left(\frac{\frac{B}{r} + r}{2}\right)^2 \quad \because A = n^2.$$

Example :— Solve $9x^2 + 52 = y^2$

$$\frac{\frac{52}{r} - r}{6} = x \text{ and } \frac{\frac{52}{r} + r}{2} = y.$$

for r we can take any number we like.

The foregoing analysis of the indeterminate equations of the first and the second degree, is marked for its lucidity that is found in any modern work on Analysis. It is the Hindus who first gave the solution of the indeterminate equations. Cantor, Kaye, Tannery and Heath see some traces of Diophantine methods in it but their views are far from facts. Even if we take into consideration some of the cases of indeterminate equations of the first degree that have been *ascribed* to Diophantus, we find that they are particular cases and can never aspire to possess the glory of *general methods* invented by the Hindus. In the cases that have been ascribed to Diophantus it would be observed that he was content with a single solution, while the Hindus have found all possible solutions. When such is the case there is no point in hesitating to record the credit to the Indians simply because they are Indians. Of all the western scholars it is only Hankel who observes that the Indians aided the general progress of Mathematics and flatly rejects the Diophantus' claim to these equations. It is Hankel who impartially observes: "the learned Brahmins of Hindustan are the real inventors of Algebra."

The invention of the *cakrawāl* or the cyclic method given by Bhāskarācārya is of remarkable quality and is nothing but the inverse cyclic method of Galois, Euler, Lagrange and the Western Mathematicians who invented again these very methods very lately. With great keenness of intellect Bhāskarācārya recognised the various solutions of the indeterminate quadratics and the European scholars have no alternative but to accept that the cyclic method given by the Hindu Mathematician constitutes the greatest invention in the theory of numbers before the time of Lagrange. It must again be pointed out that the cyclic method was given by Bhās-

karācārya in the twelfth century nearly four hundred years before it was rediscovered in Europe. In the face of all this, to deny the due credit and the honour to the Hindus, is a sheer obstinacy which only the prejudiced western scholars are capable of having. It is true that the European Mathematicians could not make use of the brilliant results in indeterminate analysis given in Hindu work and they therefore had to reinvent the same results. But the originators of this branch of science must be given their place without a grudge and the equation $Ax^2+1=y^2$ should no longer go under the name "Pell's equation" but must be legitimately designated as "Bhāskar's equation" after the name of the Mathematician who first gave its solution with all the accompanying analysis.

VEDHA

OR

Methods and Instruments of Observation of Heavenly Bodies used by the Ancient Hindus

INTRODUCTORY

If not for anything else, at least with a point of view of the history of human race in its infancy, it is worth while to study the science of Astronomy. Though most comprehensive of all sciences, Astronomy is the oldest of all as its foundations are traceable in Vedas that are as old as 2400 B.C. to 1400 B.C. In whatever manner, the European scholars may try to ascribe the origin of this science to Greece or Egypt, their attempts to do so are bound to fall flat from the various quotations from Ṛgveda that establish the fact that the ancient Indians are the originators of this science; that the Europeans have developed and perfected it to an enormous exent can never be denied, but at the same time we cannot forget to record the debt of gratitude the rest of the civilised world owes to India for the advancement the Hindus made in the early stages of the development of this science.

Unfortunately, purely astronomical works of the Vedic age are not handed down to posterity, but the sidelight thrown by the Vedas on astronomical attainments, points out that the ancient Indians were the still watchers and the scientific observers of the skies. Whether this incentive to the early cultivation of Astronomy had its origin in the religious pursuits, in the mystic splendour of the celestial scenery or in the belief that the human affairs in every kind—the life history of an individual, a state or a nation—are controlled or prearranged by the motions of the stars and the planets and that from the study of these heavenly bodies it would be possible to predict the future with certainty—or in all these reasons combined together, nobody can definitely say. But one thing is evident that the Hindus were the best observers

of the physical phenomena in the absence of optical and mechanical contrivances.

To the ancient Indians, the sky resembled a blue dome resting on the earth along the circle of horizon, the only celestial bodies observed in the day-time being the sun and occasionally the moon. At night, the sky becomes dark and the stars shoot up twinkling with diamond like gleams of light fixed on the inside of a hollow azure dome. Stars rise and set; so do the sun and the moon with her changing phases. The changes in the seasons due to the sun, the positions of the various planets and the stars, the eclipses of the sun and the moon have been carefully observed by the Ancient Indians and these observations have been noted down and piled up for generations together and have been recognised as being regular.

We live so far from these beginnings of a rational wonder that we run the risk of missing the true import of their results now so familiar to us. These are but the seeds of an important branch of culture, sown in the beginning of the investigation of this nature.

At the outset it must again be borne in mind that Vedas are not works on Astronomy. They abound in such references that fairly enable us to gauge the progress the Adhvaryus of the age have done. To the ancient Indians the influence of the sun upon the atmosphere and the soil must have made itself manifest and so, it is but natural that they must have ascribed similar influences to the moon, the planets and the stars and hence every where in Vedas we find the personification and even deification of these conspicuous luminaries. Hence we get an inter-mixture of a mathematical deduction and mythology. This leads one to believe that their scientific deductions are rather the results of pure imagination than experimentation, but it is evident that the imagination

would work only round some grain of scientific thought. For instance,

तस्माद्वा एतस्मादात्मनः आकाशः संभूतः। आकाशाद्वायुः। वायोरग्निः। अग्नेरापः। अद्भ्यः पृथिवी। पृथिव्या ओषधयः। ओषधीभ्योऽन्नम्। अन्नात्पुरुषः।

तैत्तिरीयोपनिषद् II. 1
ब्रह्मवल्ली खंडम् 1

To what extent imagination can work if at all we have to ascribe the flight of this scientific thought to pure imagination? Hence it must be admitted that whatever scientific knowledge they possessed was based upon observation aided with the imagination of the scientific mind and not of the poetic one. The above quotation gives us in a nutshell the idea the Hindus had about cosmogony and the modern science has but little departed from the theory they advanced.

The fact that the seasons are caused by the sun have been observed is evident from the following quotation:—

पूर्वामनुप्रादिशं पार्थिवानामृतून् प्राशासद्विदधावनुष्ठु।

ऋ. सं. I. 95-3

Translation:—Controlling the seasons (the sun) creates the east *consecutively* at various places.

The above sentence is pregnant with many scientific facts known at the time. In the first place, it tells us that the seasons are caused by the sun, secondly it states that the earth is round and therefore the eastern direction is indicated by the position of the sun in the morning at each and every place. Hence they definitely knew that the local time at all places cannot be the same and it depends upon the meridian of the place. Of course instead of the position of the sun on the meridian they noted its position on the horizon.

Again read the quotation:

स वा एष कदाचनास्तमेति नोदिति। यदस्तमेतीति मन्यन्तेऽह्न एव तदन्तमित्वाथात्मानं विपर्यस्यते रात्रिमेवावस्तात् कुरुतेऽहः परस्तादथ यदेनं प्रातरुदेतीति

मन्यन्ते रात्रिरेव तदन्तामित्वाथात्मानं विपर्यस्यतेऽहरेवावस्तात् । कुरुते रात्रिं परस्तात् स वा एष न कदाचन निम्रोचति । न ह वै कदाचन निम्रोचति एत स्याह सायुज्यं सरूपतामश्नुते य एव वेद ।

ऐ. ब्रा. 14-6.

Translation :—The sun never sets nor rises. When people think to themselves the sun is setting, he only changes about after reaching the end of the day and makes night below and the day to what is on the other side. Then when people think he rises in the morning, he only shifts himself after reaching the end of night and makes day below and night to what is on the other side. So really speaking he never sets at all. Whoever knows that the sun never sets, enjoys union and sameness of nature with him and abides in the same sphere.

Is this not a scientific truth based upon observation?

Units of time:—They had introduced definite units of time. It seems that the observations were recorded during a cycle of five years and so they defined 'yuga' (युग) to consist of five years. The names of the years are संवत्सर, परिवत्सर, इदावत्सर, अनुत्सर and इद्वत्सर. For rough calculations, they took the year to consist of three hundred and sixty days as is evident from the quotation:

द्वादशारं नहितज्जराय वर्वर्तिचक्रं परियामृतस्य ।
आ पुत्रा अग्ने मिथुनासो अत्र सप्तशतानि विंशतिश्चतस्थुः ।

ऋ. स. I, 164-11.

Traslation :—The sun's wheel with twelve spokes revolves round (the earth) and is never destroyed. Oh fire on this wheel are mounted (360) couples containing seven hundred and twenty people. The above language is rather figurative. The day and night are described as husband and wife and they are fitted on the wheel of the year.

Though the year was taken for ordinary calculations to consist of 360 days, but actually it consisted of 354 days

i. e. they knew that twelve lunations means 354 days approximately. For here we have

षड्हैर्मासात्संपाद्याहरूत्सृजन्ति ।

meaning thereby that six days are to be omitted in all, one day being omitted before the completion of two months i. e. 59 days make two months. If these days are not omitted

यथा वै दृतिराध्मातः एवँ संवत्सरो नुत्सृष्टः ।

तांड्य ब्राह्मण V. 10–2.

the year would unnecessarily inflate like a full blown pair of bellows.

Hence it is evident that for ordinary purposes they had their year of 360 days but for astronomical calculations, they took the year to consist of 12 lunations i. e. 354 days. So in order that the two should be in conformity of the observations taken, they introduced the अधिकमास or the intercalary month. For while giving the names of the months there appear to be thirteen names though they have definitely stated that twelve months make a year. Read the following quotation :

मधुश्च माधवश्च शुक्रश्च शुचिश्च नभश्च नभस्यश्चेषश्चोर्जश्च सहश्च सहस्यश्च तपश्च तपस्यश्चोपयाम गृहीतोऽसि सँसर्पोश्च ऽहस्पति वा

Thus the name of the 13th month is सँसर्प and it was known as अहंस्पति.

The whole year was divided into six parts and each part was known as ऋतु. Thus two months make one ऋतु.

मधुश्च माधवश्च वासंतिकावृतू । शुक्रश्चशुचिश्च ग्रैष्मावृत् ।
नभश्च नभस्यश्च वार्षिकावृत् । इषश्चोर्जश्च शारदावृत् ।
सहश्च सहस्यश्च हैमन्तिकावृत् । तपश्च तपस्यश्च शैशिरावृत ।

तै. सं. IV 4–11.

This gives which months form which ऋतु. The names of the ऋतु are (1) वसंत (2) ग्रीष्म (3) वर्षा (4) शरद् (5) हेमन्त and (6) शिशिर.

पूर्वपक्ष is used to denote the part of the month in which the moon waxes and अपरपक्ष means that part of the month in which the moon wanes. These words clearly indicate that the month ended in the new moon day.

In Taittirīya Brāhmaṇa (III. 10–10–1, 1 1, 2 etc.) we find the names of the days and nights in पूर्वपक्ष and अपरपक्ष.

Again in Taittirīya Saṁhitā (III. 4. 7. 1) we find the following

सूर्यरश्मिश्चंद्रमा गंधर्वः ।

which tells us that moon possesses the rays of the sun i. e. moon possesses borrowed light. Again the quotation

चंद्रमा अमावास्यामादित्यमनुप्रविशति ... आदित्याद्वैचंद्रमाजायते ।

shows that the phases of the moon are caused by the sun. On the new moon day the moon enters the sun i. e. the elongation of the moon is zero and therefore the phase is zero. In Ṛgveda (II 32–8) some of the phases of the moon such as अनुमति (gibbous moon) राका (full moon) कुहु or गुंगु (the new moon) सिनिवाली (the first thin crescent preceding or following new moon) are personified. These are the phases of the moon at various elongations. Thus we can aptly conclude that the movements of the moon were studied with accuracy.

Again to Hindus goes the credit of the division of the zodiac into twenty seven equal parts of 13^{0} 20′ each; and each part is known as the नक्षत्र or the constellation. The word नक्षत्र means a lunar mansion and in Ṛgveda (I 50–2) we find that the moon is personified as a bouyant youth who visits the twenty seven mansions of his spouses in turn. The names of the constellations are also found in Taittirīya Saṁhitā (IV 4–10) and so we are perfectly justified in infering that the movements of the moon in the zodiac were studied completely and its use was made as the time measurer and the month-measurer at least as early as 1400 B. C. according to the general

estimate by Sir Monier Williams (see M. Williams book Indian Wisdom). Such being the case, it is really a waste of efforts on the part of Prof. Whitney to try to show that the signs of the zodiac and much of astronomy contained above were borrowed from the Greeks. Greeks in fact had little knowledge of the astral bodies when Hindus had made enormous strides in the science of Astronomy. Prof. Whitney in his "Oriental Linguistic Studies" Vol. ii, pp. 341–421 has tried to take a review, in the face of above direct evidence, of the various opinions and arguments as to the origin of this idea and tries to convince that the original idea might be due to Greeks, Arabs or Chinese but Indian, and conveniently concludes afterwards that the matter is unsettled. However shrewd this attempt may be, it cannot be denied from the quotations cited above that it is the Hindus who are the originators of this idea. To make such "a sweeping assertion that the solar signs of the zodiac and much of the later astronomy with many astronomical terms were borrowed from the Greeks" without any direct evidence would hardly convince any one.

Mr. Monier Williams rightly observes that "the names of the Indian months have certainly been taken from the asterisms in which the moon was supposed to be full at different times of the year and what is still more significant is that the names of the lunar asterisms have clearly been derived from ancient deities". So it would be erroneous and foolish to consider that Hindus owed their knowledge of astronomy to the Greeks. It is perhaps the latter who owe a debt of gratitude to the former for supplying them much of the advanced theory Hindus developed till then.

Again in Ṛgveda Saṁhitā (V 40) there is a reference to the eclipse of the sun:

यत्वा सूर्य स्वर्भानुस्तमसा विध्यदासुरः। अक्षेत्र विदथामुग्धो भुवनान्यदीधयुः॥ स्वर्भानोरधयदींद्र मायाऽवो दिवो वर्तमाना अवाहन्॥ गुळ्हं सूर्यं तमसापव्रतेन तुरीयेण ब्रह्मणाऽविंददत्रिः॥ मामामिमं तव संतमत्र इरस्या द्रुग्धो मियसा निगारीत्॥ त्वं मित्रो असि सत्यराधास्तौ मेहावतं वरुणश्च राजा॥ ग्राव्णो ब्रह्मा युयुजानः सपर्यन् कीरिणा देवान्नमसोपशिक्षन्॥ अत्रिः सूर्यस्य दिविचक्षुराधात् स्वर्भानोरपमाया अधुक्षत्॥ यं वै सूर्यं स्वर्भानुस्तमसा विध्यदासुरः॥ अत्रयस्तमन्वविदन्नह्यऽन्ये अशक्नुवन्॥.

ऋ. सं. 5–40

Translation:—Oh sun when you were covered (eclipsed) with (स्वर्भानु) darkness by the Asuras, the whole world was plunged in darkness and people were at a loss to know their locations.

Oh God Indra, destroy the darkness on the earth.

Atri got the sun back by means of तुरीय ब्रह्मन् when he was covered with 'Apavrat' darkness.

Oh Atri let not that Asur swallow me with the desire for food. Please protect me with the help of Varuṇa.

Atri after worshipping and offering prayers was able to remove darkness and find the sun (i. e. free from eclipse). Thus Atri got back the sun that was covered with अपव्रत darkness. Only Atris can do it not others.

The above description throws an amount of light on the astronomical knowledge of the time. People were not afraid of the eclipse as a calamity, they knew it to be a phenomenon governed by the laws of nature and so did not attach any superstition to it. Atri or Atris were the mathematicians of the day who calculated the duration of the eclipse by means of observations taken with the help of तुरीय यन्त्र (known as तुरिय ब्रह्मन् after the name of the Adhvaryu who prepared it). The reference of स्वर्भानु and अपव्रत the two kinds of darkness eclipsing the sun

seems to refer to the total and partial eclipse of the sun.

Thus it would be observed that in the absence of any purely astronomical work, it would be difficult to gauge the progress made by the Hindus in this science during the period when Vedas were composed; but the passing references culled from various sections of the Vedas throw an amount of light on the knowledge of the astral bodies the learned Hindus of the time possessed. In support of this number of quotations are found in Vedas and above are given but only the choice ones omitting the others as they express the same facts already stated.

Hence we can say that Hindus of the Vedic age had definite units of time. They studied the movements of the moon by noting down its path through the twenty-seven constellation. They introduced the intercalary month in order to establish a connection between the civil year and the astronomical year. They noted down that twelve lunations means 354 days and adjusted their calendar accordingly. They had observed that the sun is on the northern side for six months and southern side for six months i. e. one year.

I must not however permit myself to continue these reflections any further the main purpose of the following pages is to bring about some salient features of the past that would fairly enable the reader to understand the methods and instruments of observation of astral bodies used by the Hindus from the Vedic age to the end of 17th century. This I have divided into four sections A, B, C and D. This being done, the lessons and conclusions to be drawn from the study of these features must be allowed to remain over for consideration and discussion elsewhere.

(A)

VEDIC AGE

The amazing accuracy of modern Astronomy is almost entirely due to our enormously increased power of laying sights. The invention of telescope and spectroscope have ushered in discovery after discovery; little wonder then that India lost her position as the one and the only one country that led the whole world in the science of Astronomy as the Indian methods of observation were based upon the naked eye in the absence of optical and mechanical contrivances. It is no exaggeration to say that the telescope has increased the powers of observation to several hundredfolds and that is why the Europeans have developed and perfected this branch of science considerably.

But before the discovery of the telescope—i. e. in the pretelescopic days, the methods of observation employed and the instruments used by the Indians to probe into the mysteries of the universe by the unaided eye, are most praiseworthy. They paved the way for further developments of the science by introducing various methods of calculations for various problems, so that with the introduction of the telescope the westerners were able to progress in the science by leaps and bounds.

We live amidst a civilisation where Astronomy is highly advanced and thus we run the risk of missing the true import of the early researches and investigations now so familiar to us; but from the point of view of the history of Astronomy these early results obtained by Hindus are but the first milestones in the progress of the

science, and what amazes us most is the sufficiently high degree of accuracy of their observation with the unaided eye. Hence from the historical point of view it is worthwhile to know what instruments were employed by the ancient Hindus to study the positions of the astral bodies.

The earliest reference of any such instruments employed by the Indians is that of Turīya instrument तुरीय यन्त्र which seems to have been made use of, to obtain the altitudes of the heavenly bodies. Ṛgveda Saṁhitā V 40 already mentioned refers to an eclipse of the sun. The 6th Śloka in it, states

गुळ्हं सूर्यं तमसापव्रतेन तुरीयेण ब्रह्मणाऽविंददत्रिः ।

which when literally translated means "Atri got back the sun covered by the shadow, by means of तुरीय ब्रह्मन्." The instrument Turīya appears to have been named after a Brahmin by name Turīya.

This instrument consists of a quadrant circle with a movable tube passing through the centre and resting on the arc of the quadrant which is graduated in degrees and minutes. Through the tube (नलिका) the heavenly bodies are observed and the angle made by the tube with the verticle gives the zenith distance—called नतांश and its complement उन्नतांश or altitude.

Of course in the absence of purely astronomical work of the Vedic age i. e, 3000 B. C. it would not be possible to know what other instruments there were or what was the form of even this तुरीय यन्त्र when Atri the mathematician of the day calculated the solar eclipse and predicted its time. But from the later developments in the observations, this Turīya instrument has been employed and it would not be bold to assert that it is handed down to Indians from the Vedic days.

In Vedas we find a number of references regarding the positions of the heavenly bodies, but as there are no allusions to the instruments they made use of, we cannot depend upon our conjectures alone.

After the Vedic age we come to the age of Vedāṅgas (1500 B. C. to 750 B. C.) This period is characterised in the Hindu civilisation as the period of sacrifices offered to Gods. During this period (as even now) the whole of India was under the powerful sway of religion and the necessity of performing the sacrificial rites to please the gods, so that they should deserve the boons which the gods confer on in return, led the Hindus of the day to mark the proper time or rather to choose the auspcious time, to commence the sacrifice. This necessitated the marking of the motions of the sun and the moon and this led to the division of the zodiac into twenty-seven parts. The moon circuits the stars in 27·32 days so that 27 is the nearest whole number and that is why the zodiac is divided into 27 parts, each part being called the Naṣkatra —lunar mansion—or the asterism so that each asterism was made to agree more nearly with the moon's mean daily motion.

During this particular age, the Vedāṅga Jyotiṣ—a work on Astronomy was composed of which only forty-nine stanzas are available. But they give us the detailed development of Astronomy of the time and since the determination of time was the main problem of the day to the Hindus, they invented the waterclock—the instrument for measurement of time.

Waterclock is simply a vessal with a small orifice at the bottom through which water flows out (when full) in the unit of time called *nāḍikā* (नाडिका). It is equivalent to twentyfour minutes. From the 24th stanza of the

Yājuṣ Jyotiṣ, it is clear that the measures they defined, are as tabulated below.

पलानि पञ्चाशतपां धृतानि
तदाढकं द्रोणमतः प्रमेयम् ।
त्रिभिर्विहीनं कुडवैस्तु कार्यम्
तन्नाडिकायास्तु भवेत् प्रमाणम् ॥ २४ ॥ याजुषज्योतिषम्

50 palas = 1 āḍhaka
4 āḍhaka = 1 droṇa = 200 palas
4 prastha = 1 āḍhaka ∴ prastha = 12·5 palas
4 kuḍava = 1 prastha ∴ kuḍava = 3·125 palas
3 kuḍavaas = 9·375 palas.

The seventh stanzas of the Ārc Jyotiṣ

धर्मवृद्धिरपांप्रस्थः क्षपान्हासः उदग्गतौ ।
दक्षिणेतौ विपर्यासः षण्मुहूर्तायनेन तु ॥ ७ ॥ आर्च ज्योतिष ।

gives: the day or night increases or decreases by a prastha in one Ayana i.e. half year—the time the sun takes to travel from one solstice to the other solstice.

From above

$$\begin{aligned} \text{nāḍikā} &= 1 \text{ droṇa}—3 \text{ kuḍavas} \\ &= (200—9{\cdot}375) \text{ palas} \\ &= 190{\cdot}625 \text{ palas} \\ &= \frac{195{\cdot}625}{12{\cdot}5} \text{ prastha} \\ &= 15{\cdot}25 \text{ prastha} \end{aligned}$$

2 Nāḍikas = 1 Muhūrta ∴ 6 muhūrta = 12 *nāḍikās*
∴ 15·25 × 12 = 183·00 days in one Ayana

One year = 366 days.

Hence the length of the year was taken to be 366 days—a fairly accurate result when we take into account the means at their disposal.

Later on we find that the waterclock is frequently made use of, for the determination of time by ancient Hindus and in Sūryasiddhānta the most celebrated work on Astronomy, we find a detailed description of this instrument viz :—

ताम्रपत्रमधश्छिद्रं न्यस्तं कुण्डे मलाम्भसि ।
षष्टिर्मज्जत्यहोरात्रे स्फुटयन्त्रकपालकम् ॥ २३ ॥

XIII Sūryasiddhānta

Translation:—A copper vessal in the form of a hemisphere with an orifice at the bottom when placed in clear limpid water sinks sixty times during the interval of day and night.

The sixtieth part of the day is named *Ghaṭikā* from the word *Ghaṭ* meaning a vessal or is also named Nāḍikā or Nāḍi 'a reed' which was used in the primitive days.

(B)

SŪRYASIDDHANTA

In Sūryasiddhānta there is a chapter devoted to the description of various instruments used by the Ancient Hindus upto the composition of Sūryasiddhānta. These instruments must have been in use for a number of years before their description was included in this work. From the results obtained in the days of Vedāṅgas as given by Vedāṅga Jyotiṣ, it is clear that the Ancient Hindus must have some types of instruments to take observations of the heavenly bodies; otherwise their calculations would not have been correct to so fair a degree of accuracy. However crude these instruments may be, they must be there and chapter XIII of the Sūryasiddhānta must have been an improvement upon an old treatise of the ancient times.

This chapter on instruments give the construction of an armillary sphere, a Śaṅkuyantra or gnomon, the Cakrayantra or the graduated wheel, Kapāl Yantra or hollow hemisphere, Dhanus Yantra or semicircle, Narayantra, Mayūr Yantra, Vānar Yantra etc.

Colebrooke in his essay on Indian and Arabian divisions of the zodiac, (Asiatic Researches IX 323, Essays ii 321 etc.) has tried to compare these instruments with those of somewhat similar nature employed by the Greeks and Arabs, and tries to throw a hint that the constructions of these instruments may have been borrowed from the instruments of western nations, but such comparisons establish, on the other hand, the fact that the Greeks borrowed their knowledge of Astronomy and Mathematics from Arabs who in their turn learnt it from

the Indians. The description of the armillary sphere appears to be so *'cumbersome a structure' to each and every European scholar from Colebrooke Davies, Burgess down to Kaye, so much so that everyone of them, has opined that its construction is an impossibility. But when we observe the massive masonry instruments at Delhi and Jaipur Observatories called the Jaiprakāsh Yantra, named, after himself by Jaisingh, we see that it is not at all an impossibility; this 'cumbersome structure' on the other hand, is the height of ingenuity of the Hindu Astronomers. The Jaiprakāsh Yantra is based upon the construction of an armillary sphere given by Sūrya-siddhānta. Arabs and Greeks never made use of an armillary sphere described in Sūryasiddhānta and in the face of such glaring historical facts to try to conclude without authentic reasons, that the Hindus borrowed the constructions of their instruments from the Greeks or Arabs, is bankruptcy of reason and logic. But such a controversy is not the object of the present work, so let us try to understand the constructions of the various instruments given in the Sūryasiddhānta.

First is the description of an armillary sphere but we would come to that later on. Let us begin with the Śaṅku Yantra or the gnomon.

Śaṅku Yantra

The construction of this instrument is given at full length in the beginning of the third chapter of the Sūrya-siddhānta. Chapter III verses 1 to 7 are :—

शिलातलेम्बु संशुद्धे वज्रलेपेऽपिवासमे ।
तत्र शाकंगुलैरिष्टैः सममण्डलमालिखेत् ॥ १ ॥

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* These are the words used by Burgess. See Burgess Sūryasiddhānta Chapters XIII,

तन्मध्ये स्थापयेत्छंकुं कल्पनाद्वादशांगुलम् ।
तत्छायाग्रं स्पृशेद्यत्र वृत्ते पूर्वापरार्धयोः ॥ २ ॥

तत्र बिंदु विधायोभौ वृत्ते पूर्वापराभिधौ ।
तन्मध्ये तिमिना रेखा कर्तव्या दक्षिणोत्तरा ॥ ३ ॥

याम्योत्तर दिशोर्मध्ये तिमिनापूर्वपश्चिमा ।
दिङ्मध्य मत्स्यैः संसाध्य विदिशस्तद्वदेवहि ॥ ४ ॥

चतुरस्रं बहिः कुर्यात् सूत्रैर्मध्याविनिर्गतैः ।
भुजसूत्रांगुलैस्तत्र दत्तैरिष्टप्रभास्मृताः ॥ ५ ॥

प्राक्पश्चिमाश्रिता रेखा प्रोच्यते सममण्डलम् ।
उन्मण्डलं च विषुवन्मण्डलं परिकीर्त्यते ॥ ६ ॥

रेखाप्राच्यपरासाध्य विषुवद्वाग्रगा तथा ।
इष्टच्छाया विषुवतो मध्यमग्राऽभिधीयते ॥ ७ ॥

Translation:—(1) On a stony surface made water level or upon hard plaster made level, draw an even circle of any radius desired.

(2) At its centre erect a gnomon of twelve units and where the extremity of the shadow touches the circle in the former and latter part of the day (पूर्व and अपर).

(3) There fixing two points on the circle, call them forenoon and afternoon points, draw between them north-south line.

(4) Draw by a fish figure an east-west line and the intermediate directions.

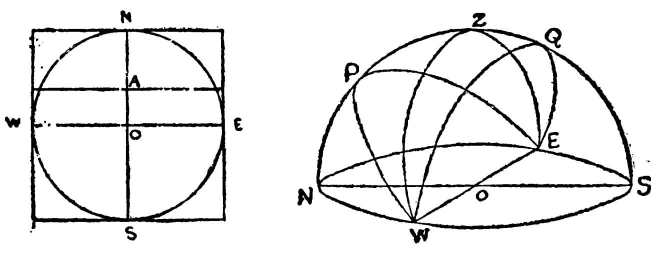
(5) Draw a circumscribing square by means of the lines going out of the centre by the digits of the base lines projected upon them is any given shadow reckoned.

(6) The east-west line is called the prime-vertical (सममण्डलम्) the equator (विषुवमण्डलम्) and the great circle passing through the pole and the east-west points, (उन्मण्डलम्).

(7) Draw likewise a line parallel to the east-west line through the end of the equinoctial shadow. The interval between any given shadow and the line of the equinoctial shadow is denominated the measure of the amplitude (अग्रा).

The above stanzas give us the following construction:

With O as centre describe a circle with any radius on level ground. At O erect a gnomon; go on marking forenoon and afternoon points on the circle and thus determine the north-south line NS. The perpendicular bisector of NS is the east-west line. (Curiously enough the construction of drawing a perpendicular of a line given in the Śulva Sūtra of Baudhāyan is made use of here). On the equinoctial day the shadow of the gnomon would fall on the line NS; it is OA say when the sun is on the meridian. Through A the end of the shadow draw a line parallel to WE. This line is called Agrā as it passes through the Agra (अग्र) or the extremity of the shadow.



The level ground is the horizon and the various circles are shown in the second figure. P is the pole and Z is the zenith in the celestial sphere in the second figure at any place, N, W, S and E being the north west south and east point of that place. Then

the circle EZW is the prime vertical सममण्डलम्
the circle EQW is the celestial equator विषुवमण्डलम्
the circle EPW is the उन्मण्डलम्

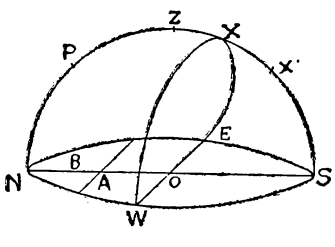
the last circle enables us to know by how far the pole is raised from the horizon of any place. This last circle is not at all mentioned in any European Astronomy and hence there is no corresponding English term for this circle.

All these circles pass through the line WE that is what the stanza 6 gives.

The use of the instrument is dealt at full length in the same chapter.

On the Viṣuvan day (i. e. equinoctial day) the shadow of the gnomon would fall along NS when the sun is on the meridian. Let ϕ be the latitude of the place and h the height of the gnomon. Then the length of the shadow is $h \tan \phi$ on that day. Let X^1 be the position of the sun on the meridian on any other day and let B be the end of the shadow of the gnomon when the sun is at X^1. Then

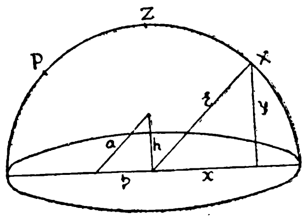
$$PZ = 90° - \phi, \quad PX^1 = 90° - \delta$$
$$ZX^1 = PX^1 - PZ = 90° - \delta - 90° + \phi = \phi - \delta.$$



The shadow $OB = h \tan(\phi - \delta)$

$AB = OB - OA = [h \tan(\phi - \delta) - h \tan \phi]$.

Again let p be the length of the equinoctial shadow a the hypotenuse of the triangle with h and p as sides and let X be the position of the sun on the meridian. Then from similar triangles in the figure.



$$\frac{h}{a} = \frac{y}{r} \text{ and } \frac{p}{a} = \frac{x}{r}$$

$$\therefore \quad \left(\frac{h}{a}\right) \text{ or } \left(\frac{y}{r}\right) \text{ gives Sin (Colatitude)}$$

$$\left(\frac{p}{a}\right) \text{ or } \left(\frac{x}{r}\right) \text{ gives Sin (latitude)}$$

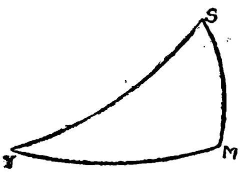
a can be obtained from $a^2 = h^2 + p^2$.

$h = 12$ units and p can be measured directly and thus the latitude of the place can be determined; the meridian zenith distance of the sun is also easily determined by simply noting down the midday shadow. The principle employed is simply that of similarity of triangles.

$\left(\frac{p}{a}\right)$ is always the Sine of the zenith distance of the sun.

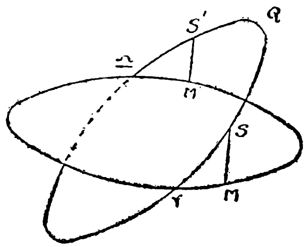
Again on the equinoctial day $\delta = \phi \pm z$. according as the sun is between the zenith and the pole or beyond the pole and the zenith. When ϕ is known, z on that day is $\text{Sin}^{-1} \frac{p}{a}$ or $\text{Sin}^{-1} \frac{p}{\sqrt{h^2 + p^2}}$ and thus the declination of the sun on that day can be obtained. It must be noted that the Hindus turned all the trigonometrical ratios into arcs.

All these results are stated as rules in verses 13 to 18. In addition verses 18 and 19 give us the rules for finding the sun's longitude on any day of the year. When the sun is at the solstice the declination of the sun is equal to the obliquity of the ecliptic. Hence if ω be the obliquity of the ecliptic and L the longitude of the sun the verse 18 tells us that the sine of the declination divided by Sin ω gives us the sine of the longitude of the sun when the sun is between Aries and the summer solstice, while verse 19 tells us that when the sun is between summer solstice and Libra this result is the supplement of the longitude of the sun i. e. $\pi - L$. The exactness of this rule can be verified for the right angled spherical triangles γ SM and Ω S'M'.



$$\text{Here } \operatorname{Sin}\omega = \frac{\operatorname{Sin}\delta}{\operatorname{Sin} L} \qquad \gamma Q = 90^\circ$$

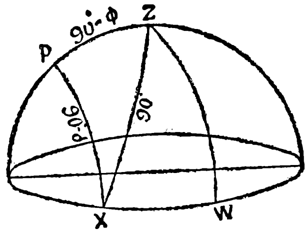
$$\therefore \quad \operatorname{Sin} L = \frac{\operatorname{Sin}\delta}{\operatorname{Sin}\omega}$$



Commenting on this verse Burgess says "It is hardly necessary to remark that this is very rough process for ascertaining the sun's longitude and could give, especially in the hands of Hindu Astronomers results only distantly approaching the accuracy".* One fails to understand what led Burgess to make such a misleading or rather utterly

* Sūryasiddhānta by Burgess.

wrong assertion, when the result stated is mathematically accurate according to the modern methods. Either Burgess was not able to grasp the fundamentals of Astronomy or he inadvertently starts with the assumption that all Hindu works are trash. No other conclusion can be drawn when we see the validity of the results given. Again if X be the position of the sun at the time of setting.



and W be the west point the arc XW was called agrā (अग्रा). PZX = 90° − XW. From the spherical triangle PZX

$$\sin\delta = \cos\phi\cos(90^\circ - XW)$$

$$= \cos\phi\sin XW$$

$$\therefore\ \sin XW = \frac{\sin\delta}{\cos\phi} = \frac{\sin\delta}{\left(\frac{h}{a}\right)} = \frac{a}{h}\sin\delta.$$

Verse 22 gives us exactly the same rule viz. the sine of the declination multiplied by the equinoctial hypotenuse divided by the gnomon sine, gives us the sine of the sun's *agrā*.

A number of results are thus obtained by taking observations of the shadow of the gnomon and are stated as rules throughout this chapter for the use of future astronomers. Thus finding the latitude of the place, the zenith distance of the sun, at any time, the longitude of the sun, the declination of the sun, the arcual distance of the sun from W the western point at the time of setting,

finding the lengths of day and night and various other problems involved therein together with their converses are worked out and stated as rules by simply observing the shadow of the gnomon.

The verse 24 of the chapter XIII states

नर यन्त्र तथा साधु दिवा च विमले रवौ ।
छाया संसाधनैः प्रोक्तं कालसाधनमुत्तमम् ॥ २४ ॥

Translation :—The gnomon instrument is very useful by day when the sun is clear and is an excellent means of ascertaining time by noting the length of the shadow.

In all the observatories of India, we find the gnomon or the Śaṅku Yantra erected according to the instructions of the Sūryasiddhānta.

There are other instruments of observation mentioned in this chapter for measurement of time but their construction is not given here, as some of them are simple and some of them have been superseded by the Śaṅku Yantra. Of these the circle, the semicircle and a quadrant (i. e. Turīya Yantra already mentioned) are made use of, to observe the altitude and the azimuth of any heavenly body.

The circle is marked with 360 degrees on its outer rim and is suspended by means of a string or chain so that the plane of the circle remains vertical. The arcual distance of the body from the point of suspension gives the zenith distance and its complement, the altitude of the heavenly body in question. Generally the circle is first suspended in the vertical plane passing through the north and south and thus the rotation of the circle from the north point to the plane of observation measures the azimuth of the body. Instead of the circle, a semicircle or a quadrant of a circle can also be made use of, for the same purpose.

The main instrument that has been described in this chapter is the construction of an armillary sphere to which a reference has already been made. It is called भूभगोल, the sphere representing the positions of the various asterisms together with the diurnal motion of the sun, the moon and the planets. Its construction is given in verses 3 to 15 of this chapter : They are as follows :—

भूभगोलस्य रचनां कुर्यादाश्चर्यकारिणीम् ॥
अभीष्टं पृथिवीगोलं कारयित्वातुदारवम् ॥ ३ ॥
दण्डं तन्मध्यगं मेरोरुभयत्र विनिर्गतम् ।
आधारकक्षा द्वितीयं कक्षा वैषुवती तथा ॥ ४ ॥
भगणांशांगुलैः कार्याः दलितैस्तिस्र एव ताः ।
स्वाहोरात्रार्ध कर्णैश्च तत्प्रमाणानुमानतः ॥ ५ ॥
क्रांतिविक्षेपभागैश्च दलितैः दक्षिणोत्तरैः ।
स्वैः स्वैरपक्रमैस्तिस्रो मेषादीनामपक्रमात् ॥ ६ ॥
कक्षाः प्रकल्पयेत्ताश्च कर्कादीनां विपर्ययात् ।
तद्वत्तिस्रस्तुलादीनां मृगादीनां विलोमतः ॥ ७ ॥
याम्यगोलाश्रिताः कार्याः कक्षाधाराद्वयोरपि ।
याम्योदग्गोलसंस्थानां भनामाभिजतस्तथा ॥ ८ ॥
सप्तर्षीणामगस्त्यस्य ब्रह्मादीनां च कल्पयेत् ।
मध्ये वैषुवतीकक्षा सर्वेषामेव संस्थिता ॥ ९ ॥
तदाधारयुतेरूर्ध्वमयने विषुवद्वयम् ।
विषुवस्थानतो भागैः स्पष्टैर्भगणसञ्चरात् ॥ १० ॥
अयनादयनं चैव कक्षा तिर्यक्तथापरा ।
क्रान्तिसंज्ञातया सूर्यः सदापर्येति भासयन् ॥ ११ ॥
चन्द्राद्याश्च स्वकैः पातैरपमण्डलमाश्रितैः ।
ततोऽपकृष्टा दृश्यन्ते विक्षेपान्तेष्वपक्रमात् ॥ १२ ॥
उदयक्षितिजे लग्नमस्तं गच्छति तद्वशात् ।
लङ्कोदयैर्यथासिद्धं खमध्योपरिमध्यमम् ॥ १३ ॥
मध्याक्षितिजयोर्मध्ये याज्यासान्त्याभिधीयते ।
ज्ञेयश्चरदलज्या च विषुवत्क्षितिजान्तरम् ॥ १४ ॥
कृत्योस्वकं स्थानं च मध्ये क्षितिज मण्डलम् ॥ १५ ॥

Translation :—Having fashioned a terrestrial of any desired size, fix a staff passing through its centre and projecting on both sides to represent the axis (मेरु).

(Fix) two circles called the ādhārakakṣā i. e. the supporting circles (answering to the colures) as also the circle of equinoxes.

Let the three circles marked with the number of degrees in the twelve signs be prepared (to represent diurnal circles at the ends of three signs Aries, Taurus and Gemini) with the radii as adopted to the scale established for those other circles in proportion to the equinoctial.

Let him (the astronomer) fix the three circles of Aries and other signs respectively (on the two supporting circles) marked with the degrees of declination north and south, at the end of respective declinations (north of the equinoxes at the ends of the said signs). The same circles answer contrariwise to the (three signs) Cancer and others. In the same manner let him fix three circles in the southern hemisphere for Libra and others and (three are) contrariwise for Capricorn and the rest. Let him also fix circles of both the supporting circles of the principal stars of asterisms in both hemispheres as also for Abhijit (and Lyræ) and for seven great sages (i. e. the seven stars forming the Ursa Major) of Agastya (Canopus) Brahma (Aurigæ) and others. In the middle of all (these circles) fix the circle of equator.

Let the two solstices and the two equinoxes be marked above the points of intersection of the supporting circles; then from equinoxes at the exact degrees of every sign (i. e. at every 30°) the places of Aries and other signs should be determined by transverse strings; another

oblique circle likewise passes through the solstices (i. e. solsticial colure).

And there is another circle called the ecliptic; in this the sun revolves always enlightening the worlds.

The moon and the other planets are attracted from the ecliptic by their nodes situated in the ecliptic and are seen (in the ecliptic) when their latitude is zero.

(The point in the ecliptic) in the eastern horizon is called the Lagna and in the western horizon it is called Aṣṭalagna.

The point of the ecliptic in the middle of the visible heaven (in the meridian) as determined through the rising periods of the signs ascertained for Lagna is called Madhyānha; the string intercepted between the meridian and the horizon is called Antya.

And the portion (of the same line) intercepted between the six o'clock line and the horizon is, equal to the sine of the ascensional difference.

Considering the given place as the highest, surround the sphere with the horizon in its middle.

(C)

Āryabhaṭṭa to Bhāskarācārya

Later on Āryabhaṭṭa in the chapter on 'Gola' or the sphere defines all the circles (great) given in the armillary sphere together with the small circles representing the diurnal motion of the sun. The verses 22–23 of the chapter on the sphere in his eminent work called Āryabhaṭīya, Āryabhaṭṭa mentions the construction of a wooden sphere. He also states how to make use of the gnomon i. e. Śaṅku Yantra and gives all the rules that are contained in the Sūryasiddhānta.

After Āryabhaṭṭa we come to the work of Brahmagupta known as Brahmasphuṭasiddhānta. In it, we find that there is a separate chapter on instruments. He has dealt with the Śaṅku Yantra, the circle, the semicircle, the quadrant, the Kapāla and the sphere. He states that there are in all seventeen instruments by which observations can be made.

Almost all the Mathematicians from Āryabhaṭṭa onwards have dealt with the instruments used for observing the astral bodies, and every Mathematician has added a separate chapter on instruments making a special mention whether a particular instrument is his own invention. Thus Bhāskarācārya, besides giving the construction of all the instruments used till his time, states that there is one more instrument that he himself has devised. In verse 16 chapter XI of Golādhyāya, he says

दृङ्मण्डलेऽत्र स्फुटकाल उक्तः सुखेननान्येर्युतितं मयातः ।

सद्गोलयुक्ते गणितस्य सारं स्पष्टं प्रवक्ष्ये फलकाख्य यन्त्रम् ॥

Translation:—As others have not ascertained with ease the apparent time by observations of altitudes in a vertical circle, I have therefore laboured myself in devising an instrument called फलक यन्त्र, the use of which I proceed to explain perspicuously. It contains in itself the essence of our calculations which are founded on the true principles of the doctrine of the sphere.

From verses 18 to 27 he gives the detailed description of the Phalaka Yantra and its use as well.

कर्तव्यं चतुरस्त्रकं सुफलकं खांकाङ्गुलैर्विस्तृतम् ।
विस्ताराद्विगुणायुतं सगुणकेनायाममध्ये तथा ॥
आधारःश्लथ श्रुंखलादिघटितः कार्या च रेखा तत-
स्त्वाधारादिव लम्बसूत्रसदृशी सा लम्बरेखोच्यते ॥ १८ ॥
लम्बं नवत्यङ्गुलकैर्विभज्य प्रत्यङ्गुलं तिर्यगतः प्रसार्य ।
सूत्राणि तत्रायतसूक्ष्मरेखा नीवाभिधानाः सुधिया विधेयाः ॥ १९ ॥
आधारतोऽधः खगणाङ्गुलेषु ज्यालंबयोगे सुषिरं च सूक्ष्मं ।
इष्टप्रभाणा सुषिरे शलाका क्षेप्याक्ष संज्ञा खलु सा प्रकल्प्या ॥ २० ॥
षष्ट्यङ्गुलव्या समतश्व रन्ध्रात्कृत्वा सुवृत्तं परिधौ तदैक्यम् ।
षष्ट्या घटीना भगणांशकैश्व प्रत्यंशकं चाम्बुपलैश्वद्दि ग्भिः ॥ २१ ॥
अग्रेसरन्ध्रा तनुपट्टिकैकाः षष्ट्यङ्गुला दीर्घतया तथांक्या ।
यत्खण्डकैः स्थूलचरं पलाद्यं तद्गोकुद्दत्स्याच्चरशिञ्जिनीह ॥ २२ ॥
वेदा भवाः शैलभुवो धृतिश्व विश्वे च बाणाः पलकर्णनिघ्नाः ।
अर्कोद्धताः स्थुः क्रमशः स्वदेशे राश्यार्द्धलम्ब्यानि हि खण्डकानि ॥ २३ ॥
तैः क्रान्तिपाताढ्य रवेर्भुजज्या षष्ट्या दूधृताक्षश्रवणेन युक्ता ।
दिघ्नी कृताप्ता भवतीह यष्टिः सापट्टिकायां सुषिरास्तदेयाः ॥ २४ ॥

धार्य्यं तथा फलकयन्त्रमिदं यथैव
तत्पार्श्वयोर्लगति तुल्यभिनस्यतेजः ।
छायाक्षजा स्पृशति तत्परिधौ यमंशं
तन्मांशके प्रतिमता तरणिः प्रकल्प्यः ॥ २५ ॥
अक्षप्रीतां रविलवगतां पट्टिकां न्यस्य तस्मात् ।
यष्टेरग्रादुपरि फलकेऽधश्व गोलक्रमेण ॥

यत्नादेयश्चरदलगुणस्तत्र याज्या तयात्र ।
छिन्ने वृत्ते तलगघटिकाः स्युर्नता लम्बकान्ता ॥ २६ ॥
लम्बाद्देया विनतघटिकास्तज्य कातश्चरज्या ।
व्यस्ता देया भवति च तथा यापरा तत्र मौर्वी ॥
धार्य्योपट्टी स्पृशति हि यथा तज्ज्यकां यष्टिचिह्नं ।
पट्टिस्थाने निपतति तदाक्षस्य नूनं प्रभास्य ॥ २७ ॥

Translation—(18) Let a clever astronomer make a board (फलक) rectangular in shape, the height being 90 aṅgulis and length 180 aṅgulis. Halve its length and at that point attach a movable chain to hold it; from the point of suspension let him draw a perpendicular.

(19) Let him divide this into 90 equal parts which will also be aṅgulis and through each part draw a line parallel to the top and bottom edges: these are called sines.

(20) At that point of the perpendicular intersected by the 30th aṅguli a small hole is to be bored and in it is to be placed a pin of any length and it is to be considered as the axis.

(21) With this as centre draw a circle (with a radius of 30 aṅgulis) of 60 aṅgulis as diameter. Mark the circumference of the circle with 60 ghaṭis and 360 degrees each degree being subdivided in 60 kalās.

(22) Let a thin paṭṭikā or the index arm with a hole at one end be made and length be 60 aṅgulis. Let it be marked properly.

The approximate ascensional difference in palas determined by the khaṇḍakas or parts being divided by 19 will here become the sine of the ascensional difference (adapted to this instrument).

(23) The numbers 4, 11, 17, 18, 13, 5 multiplied severally by the akṣakarṇa (अक्षकर्ण) and divided by 12

will be the khaṇḍakas or portions at the given place, each of these being for each 15 degrees of *bhujā* of the sun's longitude respectively.

(24) Now find the sun's true longitude by applying the precession of the equinoxes to the sun's place and adding together as many portions as correspond to the bhujā of the sun's longitude above found, divide by 60 and add the quotient to the akṣakarṇa (अक्षकर्ण). Now multiply the result by 10 and divide by 4. The quotient here is called Yaṣṭi in aṅgulis and the number thus found is marked off as the arm of the paṭṭikā counting from its hole penetrated by the axis.

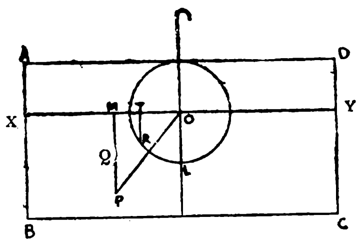
(25) Now hold the instrument so that the rays of the sun shall illuminate both of its sides (to bring it into the vertical plane of the sun). The place of the circumference marked out by the shadow of the axis is assumed to be the position of the sun's place.

(26) Now place the index arm on the axis and putting it on the sun's place from the point at the end of Yaṣṭi, set off carefully above or below (i. e. parallel to the lambarekhā) on the instrument, the sine of the ascensional difference above found, setting it above if the sun be in the northern hemisphere and below if it be in the southern hemisphere. The distance from the point where the sine which meeting the end of the sine of the ascensional difference thus set off, cuts the circle to the lowest part of the circle will represent the ghaṭis to or after midday.

(27) Set off the time from midday on the instrument counting from the lambarekhā, from the end of the sine of this time, set off the sine of the asensional difference in a line parallel to the lambarekhā but below or above according as it was set off above or below in finding the time

from the shadow (this operation being the reverse of the former). The sine met by the sine of the ascensional difference, thus set off, is the new sine across which the paṭṭikā or the index is now to be placed till the Yaṣṭi-cinha (the mark of the index) falls on it. This position will assuredly exhibit the place of the shadow of the axis.

The use of the instrument is in fact to find the hour angle of the sun at any time of the day. To understand the working see the accompanying diagram

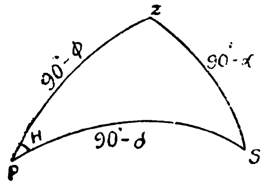


ABCD is rectangle in which AB = 90 and BC = 180. Take AX = 30 and draw XY parallel to AD. With O as centre (O being the midpoint of XY) draw a circle with 30 radius. At O put a pin perpendicular to the plane of the rectangle carrying a paṭṭikā or the index arm which can revolve about the pin along the plane of the rectangle. The paṭṭikā is marked properly.

Let S be the point on the circle where the shadow of the pin cuts the circle when the board ABCD is held in the vertical plane passing through the sun.

Let a be the altitude of the sun, δ its declination, H its hour angle and ϕ the latitude of the place. If $A = \frac{\pi}{2} + H'$ where H′ is the hour angle of the sun at the time of rising then A is known as the ascensional diffe-

rence. $(2A - \pi)$ is obviously the length of the day. Consider the Spherical triangle ZPS



$$\cos H = \frac{\sin a - \sin\phi \sin\delta}{\cos\phi \cos\delta}$$

$$= \frac{\sin a}{\cos\phi \cos\delta} - \tan\phi \tan\delta.$$

But $\cos H' = -\tan\phi \tan\delta$

and $H' = A - \frac{\pi}{2}$

$$\therefore \sin A = -\tan\phi \tan\delta$$

$$\therefore \cos H = \frac{\sin a}{\cos\phi \cos\delta} + \sin A.$$

If the declination be south, the last term would be negative. Now from the Śaṅku Yantra where the Śaṅku is 12 units in height, if h be the hypotenuse of the triangle formed by the Śaṅku and its shadow when the sun is on the meridian of an equinoctial day

$$\cos\phi = \frac{12}{h}.$$

$$\therefore \quad \cos H = \frac{h}{12} \frac{1}{\cos\delta} \sin a + \sin A.$$

Now $\frac{h}{12\cos\delta}$ is written as

$$\frac{1}{12}\cdot\frac{h}{12}\left[12 + \frac{12(1-\cos\delta)}{\cos\delta}\right]$$

$$\text{i. e. } \frac{1}{12}\left[12 + \frac{12 \text{ versin } \delta}{\cos\delta}\right]\frac{h}{12}.$$

The values of $\frac{12 \text{ versin } \delta}{\cos\delta}$ are given as $\frac{4}{60}$, $\frac{15}{60}$, $\frac{32}{60}$, $\frac{50}{60}$, $\frac{63}{60}$, $\frac{68}{60}$ respectively for the longitudes 15°, 30°, 45° ....

......90° of the sun. And the verse 23 gives the differences of the numerators of these fractions viz. 4, 11, 17, 18, 13, 5. The bar that revolves round the pin is marked to the length given by $\frac{h}{12 \cos \delta}$ and if x be its value i. e. if OP $= x$

$$\cos H = x \sin a + \sin A.$$

$$= PM + \sin A.$$

or

$$= PM - \sin A$$

according as the declination is north or south.

If further $\text{Sin } A = PQ$ then

$$\text{Cos } H = PM - PQ = MQ = RT$$

$$= \text{Cos } \angle ROL$$

$$H = \angle ROL.$$

Hence it would be observed that though the process of obtaining the hour angle appears to be cumbersome still in the absence of the knowledge of modern Spherical Trigonometry, the results obtained are quite correct.

What we have to note down throughout is that the tradition of taking observations by means of instruments is kept up all along; and whatever new instrument was invented by any Mathematician, he gave its construction to keep it a legacy for the future Mathematicians in the field.

Another instrument for finding the time to sunrise at any time of the night is invented by Padmanābha who lived in the beginning of the fourteenth century. This instrument is known as the ध्रुवभ्रमयन्त्र. It consists of a rectangular board with a horizontal slit at the top. Below the slit there are concentric circles and at the centre it

carries a pointer denoting direction as in a map. The pointer can be rotated about the centre along the plane of the board like the hands of a clock moving on its dial.

Hold the instrument in a vertical plane so that Ursæ Minoris and the pole star are both seen along the centre line of the slit. The west indicator then gives the sidereal time, the north indicator shows the rising, culminating and setting lunar asterisms and the east indicator shows the rising and culminating solar signs of the zodiac. Having observed the sign and the degree in which the sun then is, from the calender, the position of the sun can be ascertained. Placing the east indicator at this sign on the inner circle, the west indicator gives the sidereal time of sunrise. The difference between this and the sidereal time of observation from the almanac gives the time before sunrise. When the Ursa Minoris is on the eastern side of the pole star the instrument is to be held with the slit uppermost and when it is to the west of the pole star the instrument is to be inverted.

The principle of this instrument is very simple. Ursa Minoris is the brightest star in the Ursæ Minor and is roughly 17° from the pole star. The line joining these two revolve completely like all the heavens once in 23 hours and 56 minutes approximately. Hence the measurement of the angle which this line makes with the vertical gives the knowledge of the position of the star sphere and therefore of the signs. The sun's position in the signs being known from the calender, the time is easily determined by addition or subtraction. But this instrument is not important as it requires the help of the calender.

From the twelfth century onwards, almost all the Mathematicians wrote separate works on the construction and the use of instruments of observation. The most

important of them all is Yantra Cintāmaṇi यन्त्रचिन्तामणि by Cakradhar; but it would be remarked, that the instruments described therein, are not in any way improvements on these old constructions, but they have been elaborately described so that their use can be easily understood. The Mathematician Rām who has written a commentary on Yantra Cintāmaṇi remarks

विलोकितानि यन्त्राणि कृतानि बहुधा बुधैः ।
मतः शिरोमणिस्तेषां यन्त्रचिन्तामणिर्मम ॥

This clearly indicates that a number of instruments were in vogue till then, but unfortunately all the works on them are not available.

(D)

OBSERVATORIES.

In all the astronomical works of the Hindus it must be noted that Ujjain, formerly known as Avanti, is taken as a place situated on the prime meridian and tradition also makes it the centre of astronomical learning in India. Sūrya Siddhānt states.

राक्षसालय दिवौकः शैलया मध्यसूत्रगा ।
रोहितकमवन्ती च यथा सन्निहितं सरः ॥ ६२ ॥

Translation:—(The cities named) Rohitaka Ujjain, Kurukṣetra etc. are on the line between Laṅkā and the north pole of the Earth. This line is called the middle line of the Earth.

Even Pañca Siddhāntikā refers to Avanti in verse XIII as the place on the prime meridian. Bhāskarācārya mentions Ujjain in several connections as the place of zero longitude. Nowadays Greenwich meridian is taken as the meridian of zero longitude in all British nautical almanacs and in this respect may be termed as the Ujjain of the British.

Ujjain is one of the most ancient astronomical centres of the world and has been the centre of Hindu astronomical teachings. Whether at Ujjain there was any place reserved for all learned Pundits to take observations, nobody knows as there are no relics of the oldest observatory as such. Perhaps the present observatory is built at the very place where the old one was situated and it may be that the Mathematicians took their observations there, with the instruments they made and

depended more on their individual merit for the results they found. It appears that later on, there was an urge to keep a proper place with fixed instruments which should be the property of all and where anybody can go and take observations so that the error of personal equation can be eliminated to the lowest.

Such an attempt seems to have been made in the reign of Akbar. Mānsingh the rajah of Amber built MānMandir in the beginning of the 17th century. Sir Robert Barkar who visited India in the year 1749 states from the information he then got, that MānMandir observatory was built by Akbar and Mānsingh being a vassal of Akbar, it may be that the construction of the observatory is ascribed to Akbar. Whatever it is, one thing is certain that the observatory at Benares that is known as MānMandir, has its foundations laid down in the beginning of the 17th century. To this observatory after a century or so a number of instruments were added by Jai Singh, so much so that the observatory became almost new.

MānMandir observatory is not the only observatory improved by Jai Singh. About four or five miles from Amber there was in those days a plane ground to the extent of twenty or twentyfive square miles. Here it appears there existed an old observatory from the days of Mānsingh or even earlier, as the plane surface was a convenient level ground for observing the rising and the setting of the heavenly bodies: or it may be that such a place being the most suitable for observations, Jai Singh built a number of instruments of his own at that place. Later on he took so much fancy for the place that he founded the city of Jaipur around the observatory. The whole city was planned by an eminent architect Ratnākar Puṇḍarik who came from Bengal. The founda-

tion of the city however spoilt the purpose of the old observatory as the surrounding walls of the palace and the other palacial buildings make observation of the rising or setting of any heavenly body almost impossible.

Tieffenthaler—a Jesuit Missionary had travelled in India from 1743–1786. In his book "Historish Geograghishe Bescheibung von Hindustan" 1785 Vol. I, he describes.

'But a place that deserves detailed description is that where astronomical observations are made; it is such a work as is never seen in this part of the world and by the novelty and the grandeur of the instruments strikes one. This large and spacious observatory is close to the king's palace...' Then he goes on describing the important instruments one by one and then further says: 'the matter which detracts from the value of the observatory is that in a low situation surrounded by walls the observer cannot see the rising or setting of the stars'.

Such a criticism by foreign visitors of the observatory is but the representation of a truth, but that confirms the fact that the observatory situated as it is, is even older than Jai Singh: and as I have suggested probably it dated to the days of Mānsingh or even earlier.

There is another point to be noted and that is the instrument known as the *rāśivalayas*! Jai Singh who gives the use of each and every instrument is silent about these rāśivalayas, and upto 1907 nobody cared to know the meanings of those instruments. In the year 1907 Mr. Garett tried to know their meaning with the help of the local astronomers who failed to guide him properly. With good intentions, he tried to rebuild those instruments as the old ones were in a dilapidated condition. Instead of replacing them as they were, he tried to show

his intelligence by trying to establish their connection with the rising signs of the zodiac, with the result, that he changed the old instruments according to his own scheme. Thus the whole of humanity lost a monumental structure which the later Mathematicians would have found the use of. Of course total ignorance or rather the lethargy of the intelligentsia among the Indians is responsible for so tremendous a loss as that. It is a fact that till Garrett tried to find their meaning no Indian seemed to know the use of the rāśivalayas. Perhaps Jai Siṇgh too might not have been able to find out their real use and therefore kept them as they were; all our conjectures in that direction are in vain, since these instruments have been completely changed by Garrett. At the same time these rāśivalayas cannot be later additions as there is no mention of them by any Mathematician of the court of Jaipur.

Hence we can clearly conclude that the observatory at Jaipur must have been built in the days of Mānsingh if not earlier perhaps together with the MānMandir observatory at Benares and later on because Jai Singh added and improved upon them quite a lot so much so that they became literally new, they came to be known to the future generations as the observatories established by Jai Singh as he had founded the observatories of Delhi and Muthra. Delhi observatory is not the first one built by Jai Singh and it is not at Delhi that the principal observations were made which were to form the bases of his new tables—the Ziz Muhammad Shāhi. These tables were completed by recording observations for seven years and these observations must have been taken at Jaipur observatory.

All the same it is not my intention to enter into the controversies on minor points as these and connive at the main issue—viz. the instruments of the observatory and

their use. Jai Singh has done a great service by leaving us an abiding monument of the Hindu astronomy—a visible link in the history of the science—a link that offers us a clearer grasp and insight into the conditions of Hindu astronomy than any number of treatises and ancient manuscripts. Jai Singh did three things:

(1) He revised the star catalogue and reformed the calendar.

(2) He revived the Hindu astronomy by patronising the learned Pundits—Jagannāth Samrāṭa being the foremost.

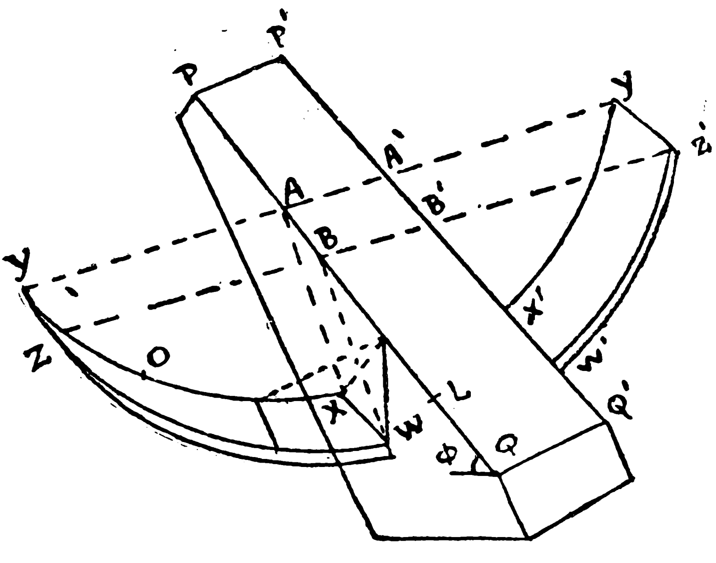
(3) He promoted the study of Mathematics.

It is true that he did not make any striking astronomical discovery, but with a view to keep the Indians in the fore-front with the march of time as well, he provided the Indian Mathematicians an opportunity to keep up the tradition of the early scholars. With this view he discarded the brass instruments and built masonry ones instead. But the discovery of the telescope and later on the spectroscope completely eclipsed the old instruments by increasing thousandfold our power of laying sight. Hence Jai Singh's methods fell into the background as they were out of date. All the same we cannot but credit him upon his labours as the last expiring efforts of the old astronomy—the astronomy of the unaided human eye. If the Hindu astronomy is to be revived again and if the Hindus are to lead the whole world in the field of astronomy as they did to the end of the 17th century, Hindus must adapt their observatories with the modern instruments and contrivances. Even these old observatories fitted with modern requirements would go a long way to improve our ideas and we would no longer be

required to look to the westerners for correcting our calendars.

Samrāṭa Jagannāth—a Mahārāshtrian—was one of the most eminent astronomers of those days and Jai Singh appointed him as his Guru. He invented the instrument known as the Samrāṭa Yantra after his own name. This instrument is useful in finding the hour angle and the declination of any star or the sun.

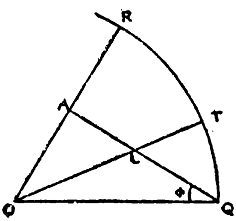
It consists of a triangular gnomon with the hypotenuse parallel to the earth's axis, with two equinoctial dials in the forms of semicircles such that on either side of the gnomon there is a quadrant and the centres of these semi-circles lie on the gnomon. The front view of this instrument is as shown in the figure.



The triangular structure PQQ'P' in the centre is in the north south direction i. e. PQ is in the plane of the

meridian and QP points towards the pole so that QP is inclined to the horizontal at an angle ϕ which denotes the latitude of the place. The two quadrants on either side of the triangular structure have their centres on PQ and P′Q′ respectively. Thus A is the centre of the arc XY and B is the centre of the arc WZ. These quadrants are graduated with degrees and minutes (kalās).

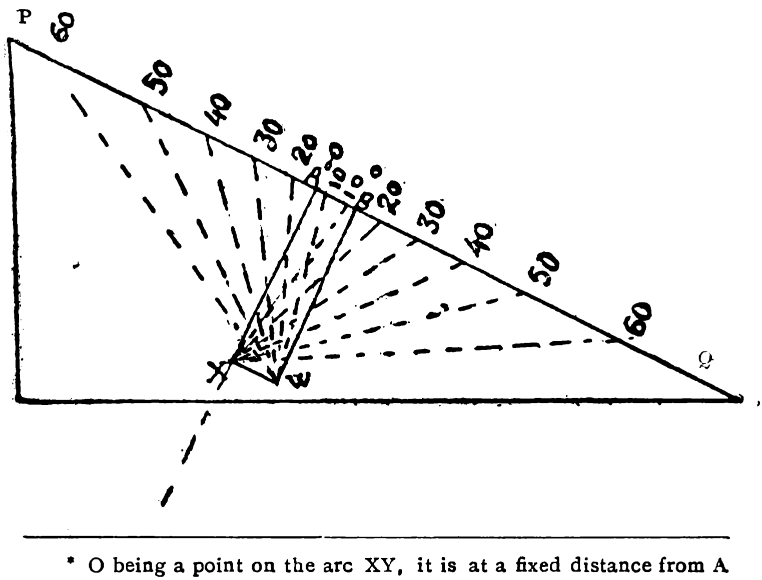
Suppose now that a particular star is on the eastern side of PQ and on the southern side of the quadrant XY. In order to know its hour angle what the observor has to do is to move his eye on the quadrant XY till his eye, a point on PQ, and the star are in one line. Let O be the position of the eye of the observer and L the point on PQ such that OL passes through the star, then the arc OX which measures the angle between the meridian plane and the plane passing through the pole and the star i. e. the plane formed by PQ and OL and therefore arc OX is the east hour angle of the star. Again the length A to Q is graduated in the following manner:



Take the plane passing through O and PQ. With O as centre and radius OQ describe a circle cutting OA produced in R. (See the above figure). The arc RQ is graduated in degrees and minutes; and then every division of it is joined to O and wherever these joins cut

the line AQ the points of intersections are marked according to the divisions of the arc RQ.

Suppose that the direction OL cuts the arc RQ in T where RT subtends an angle of say 22 degrees at O then L is marked 22 degrees and so on, then the measure of the declination of the star is 22 degrees i. e. ∠ AOL south 90+∠ AOL being the polar distance. Thus the line AQ is graduated downwards from A to Q along AQ and upwards from B to P in the same manner.* If the star be on the eastern side of PQ and the northern side of the quadrant the observations are to be made by keeping the eye on the arc WZ. Similarly if the star be on the western side of PQ the arcs X'Y' and W'Z' are to be made use of.

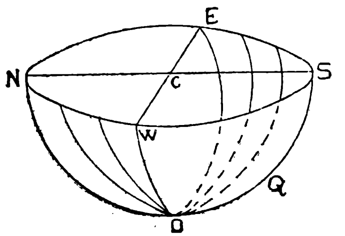


* O being a point on the arc XY, it is at a fixed distance from A the centre of the arc XY and hence these divisions on the line AQ are not affected by the change in the position of O.

If the same observations are taken on the sun, we get the sun's declination and the shadow cast by the triangular structure on the quadrants give automatically the time to or after midday i. e. the hour angle of the sun. The readings taken by this instrument are accurate to a second. Thus this instrument being most useful appears in all the observatories—viz. Delhi, Jaipur, Ujjain, Benares and Muthra. This is thus a definitely better instrument than the Phalak Yantra of Bhāskarācārya.

Another instrument of special mention is the Jai Prakāśa Yantra. It may be an invention of Jai Singh and therefore might have been named like that. But Jagannāth calls it the crest jewel यन्त्र शिरोमणि of all instruments. Its idea is suggested from the armillary sphere described in Sūrya Siddhānta and the other celebrated works. It is an armillary sphere cut into two by the plane of the horizon only the upper portion being kept, as that is the only portion visible to us. Jagannāth Samrāṭa has given the full description of the instrument.

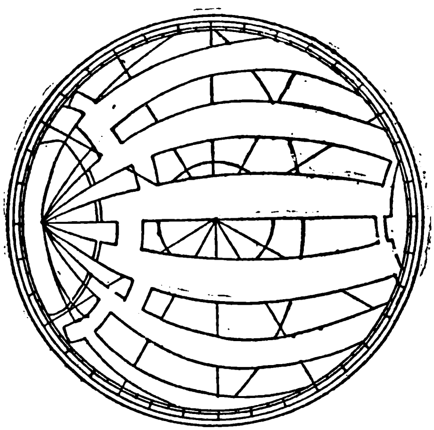
It consists of a concave hemisphere dug in the ground. The rim of the hemisphere represents the horizon and is graduated with degrees and minutes.



NQS represents the meridian passing through the south and the north points, i. e. it is the prime meridian. Mark the east and west points and join them so as to cut the north south line at C. Take the arc SQ equal to the latitude of the place and Q then becomes the south pole and the direction QC would pass through the pole of the celestial sphere. With Q as centre and spherical radii $\frac{\pi}{2} \pm \omega$ draw two circles ω being the obliquity of the ecliptic. Both these circles are to be graduated with signs, degrees and minutes. These are obviously the circles described by the poles of the ecliptic round the pole. Mark on these circles positions representing the rising of the twelve signs in a westerly direction. Taking each of these points as centre describe circles, they will be twelve in number. These circles represent twelve different positions of the sun in the ecliptic. With O the lowest point of the hemisphere as centre and spherical radii 6, 12, 18......describe fifteen circles, horizon being the 15th. Through O draw sixty circles of azimuth. EW and NS are the cross wires cutting each other at C where a small hole is kept. The sun's shadow falls through this hole on the inside of the hemisphere and revolves as the sun revolves and thus enables to note down the coordinates of the sun. In the hemisphere, paths are dug out for the observer to have ingress into, and egress out of the instrument. When the passages are cut some portion of the hemisphere will have to be taken out but this loss has been made good by having the duplicate of the instrument such that wherever there are passages in the first there would be the portion of the instrument in the second.

In this way there are five passages in each of the duplicate instruments at the observatories of Delhi and

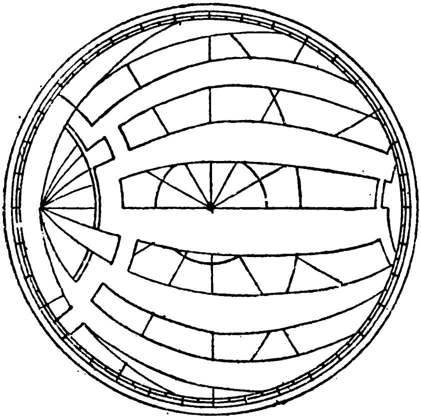
Jaipur. Not only the observations can be taken on the sun, but also on any star by placing the eye at the proper graduated point and observing the passage of the body across the north and south line i. e. the prime meridian of the place.



JAYAPRAKASH
NO 1

Commenting on this instrument Kaye observes " Jai Prakash is practically the hemisphere of Berosus somewhat elaborated". I would like to be solicitous to allow Kaye all the credit for all the labours he has taken on the work on the Indian observatories but I cannot avoid expressing here my firm conviction that as an authority on the subject of Indian Mathematics or Indian Observatories he is hardly to be trusted and his remarks must be used with extreme caution. The above statement of his is entirely false and therefore misleading. Kaye always tries, without authentic reasons, to establish that the Indians borrowed their

knowledge of Astronomy from the Greeks and the Arabs, and wants to show that Indians lack originality.



JAYAPRAKASH
NO 2

So in this case also, he makes a statement that is erroneous. Berosus*—priest of Bel at Babylon, translated into Greek the standard Babylonian work on Astrology and Astronomy and compiled the history of his country from native documents which he published in Greek in the reign of Antiochus II. Nowhere there is any mention of any instrument descovered by him. All the same Kaye finds a certain Berosus perhaps another creation of his own, and perhaps from his own brain and gives the description of the Berosus bowl. Kaye observes: 'Berosus bowl consists of a hemisphere with its rim horizontal and in the centre in the same plane as the horizontal edge was fixed a bead whose shadow on the concave surface of the hemisphere

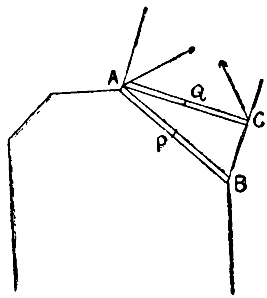
* Encyclopedia Britannica.

marked the trace of the sun's diurnal path.' Kaye further adds: 'the resemblance (of this bowl) to the Jai Prakash is striking enough but it is doubtful whether Jai Singh had any knowledge of the earlier instrument: he could only have learnt it from the Muslim Astronomers,' and further with a pretence of impartiality he says, 'the Jai Prakash is however something more than the bowl of Berosus, for it is fully graduated.' Such statements would easily misguide the reader but a little thought would prove that they are utterly worthless. Jai Prakash Yantra is mainly based upon the armillary sphere described in Sūrya Siddhānta and other celebrated works of the Mathematicians of the 4th to 15th centuries such as Āryabhaṭṭa, Brahmagupta, Bhāskarācārya and others. Jai Singh's theories, his instruments and books are assuredly not copied from European sources as there is no trace of it anywhere in his works. His tables even are perfectly original as even a glance at them would convince any layman.

There is another instrument known as the Krāntivṛtta and is useful in directly measuring the celestial latitude and longitude of any heavenly body. It consists of a masonry pillar having a north face fitted with a circular red stone in the plane of the equator. In the centre it carries a brass pin round which a metal framework ABC can easily be rotated, the angle of rotation being registered by a pointer moving over the circumference of the stone graduated with signs, degrees and minutes. The frame work consists of two brass circles AB and AC and P is the position of the pin.

$\angle BAC = \omega$ the obliquity of the ecliptic and is taken as 23° 27′ in those days. AC obviously represented the ecliptic and AB the equator. The ecliptic circle is fitted

with two quadrants at right angles to the plane AC with a bar whose centre passes through another pin at Q so that the bar can rotate about Q freely. These quadrants are fitted with sighting bars and are graduated in degrees



and minutes. They can rotate about the bar AC. The lower circle AB is also graduated. The rod AC points towards the position of the solstices and the lower radius of the quadrant makes with it a certain angle as it moves in the plane of the star and the pole of the ecliptic. This angle subtracted from or added to 90° gives us the longitude of the star and the sighting bar the latitude. At present this instrument appears in the Jaipur observatory only and there too, the two quadrants on AC have disappeared.

Another remarkable instrument is the Yantrarāj which has been in use in India from the 13th century onwards. It was originally invented by a Mathematician named 'Mahendra Sūrya' who was the astronomer royal in the days of Phiroj Shah Tughlakh. Jai Singh has also written a book on it. It contains eight verses. This instrument is more or less like a slide rule and enables us to solve a number of problems involving relations between azimuth, altitude, latitude, longitude, time and positions

of the heavenly bodies quite mechanically. The back of this instrument is fitted with a sighting bar with which only the altitudes of the heavenly bodies are to be observed. This is the only direct observation to be made and then this instrument serves as a calculator.

The principle of this instrument is based upon the stereographic projection. The stereographic projection possesses the property that all circles on the sphere project into circles except those that pass through the point of projection. The circles passing through the point of projection, project into straight lines. The construction of this instrument is given by Jai Singh. To understand it thoroughly well mathematical working has been added wherever necessary.

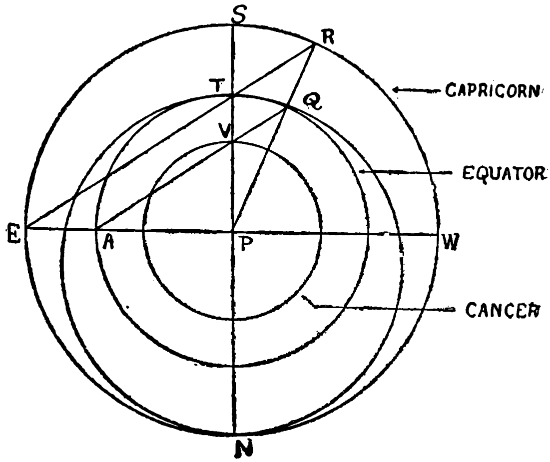
1. Take a circular plate of wood or any metal and carve on it two diameters at right angles, one vertical and the other horizontal. The uppermost point being regarded as the south point. Thus SN is the north south line and EW the east west line. Take an arc SR towards west such that it subtends an angle ω equal to the obliquity of the ecliptic. If P be the centre of the plate join PR. Join ER as well cutting PS in T. With P as centre and PT as radius describe a circle cutting PR in Q and EW in A. Join AQ cutting PS in V. With P as centre and radius PV describe a circle. The following names are given to these circles.

(i) The circle (P, PS) is called the tropic of capricorn.

(ii) The circle (P, PT) is called the equator.

(iii) The circle (P, PV) is called the tropic of cancer.

2. From the point B diametrically opposite to A measure BC = ϕ the latitude of the place towards the north in the figure (see figure on page 192). Join AC and let it cut NS in D. D is called the first point. Take an



arc AF = ϕ (= BC) so that F is diametrically opposite to C. Join AF and produce it to meet NS produced in G. G is called the second point (the position of G is not shown in the figure). On DG as diameter describe a circle and this circle is to represent the horizon.

3. Suppose we have to draw a parallel of altitude *a*. Take an arc CI towards S such that CI subtends an angle *a* at the centre. Join AI cutting NS in K. Take an arc FJ = CI and join AJ cutting NS in L. Then on LK as diameter describe a circle.

$$LK = PL - PK = AP.\ \tan P\hat{A}L - AP.\ \tan P\hat{A}K.$$

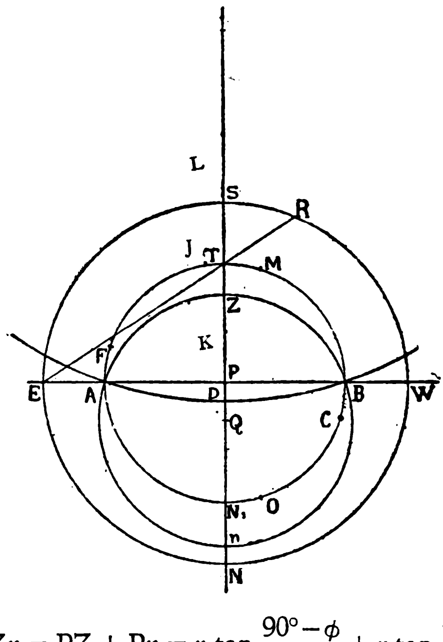
If AP = r PAL = PAJ = $\frac{1}{2}(180° - \phi - a)$ and PAK = $\frac{1}{2}$ BPJ = $\frac{1}{2}(a - \phi)$.

$$LK = r \cot \frac{a+\phi}{2} - r \tan \frac{a-\phi}{2} = \frac{r \cos a}{\sin \frac{a+\phi}{2} \cos \frac{a-\phi}{2}}$$

$$\therefore \quad \tfrac{1}{2} LK = \frac{r \cos a}{\sin a + \sin \phi}$$

The midpoint of LK is to be taken as the zenith.

4. Take an arc TM towards W such that it subtends an angle ϕ. Join AM cutting NS in Z. Let N_1, be the point where the circle (P, PT) cuts NS. Take an arc N_1 O such that it subtends an angle ϕ. Join BO and produce it to meet NS in n. Then Zn is the diameter of the prime vertical. On Zn as diameter describe a circle which is the prime vertical.



$$\text{Now } Zn = PZ + Pn = r \tan \frac{90° - \phi}{2} + r \tan \frac{90° + \phi}{2}$$

$$\therefore \quad Zn = \frac{2r}{\cos\phi} = 2r\sec\phi.$$

$\therefore$ The radius of the prime vertical is $r\sec\phi$.

If Q be the centre of the prime vertical

$$PQ = Pn - Qn = r\tan\frac{(90^\circ+\phi)}{2} - r\sec\phi$$

$$= r\left[\frac{\sin\left(45^\circ+\frac{\phi}{2}\right)}{\cos\left(45^\circ+\frac{\phi}{2}\right)} - \frac{1}{\cos\phi}\right]$$

$$= r\left[\frac{1+\tan\frac{\phi}{2}}{1-\tan\frac{\phi}{2}} - \frac{1}{\cos\phi}\right]$$

$$= r\left[\frac{\cos\frac{\phi}{2}+\sin\frac{\phi}{2}}{\cos\frac{\phi}{2}-\sin\frac{\phi}{2}} - \frac{1}{\cos\phi}\right]$$

$$= r\left[\frac{1+\sin\phi}{\cos\phi} - \frac{1}{\cos\phi}\right]$$

$$= r\tan\phi.$$

5. The lines joining the points of intersection of the horizon and NS to the divisions of the equator would cut the prime vertical in points which are to be taken as the graduations of the prime vertical, since the declination of the zenith is the latitude of the place.

6. After graduating the horizon, draw circles passing through each graduation of the horizon, the zenith and the nadir. The centres of all such circles would obviously lie on the line passing through Q and parallel to the east west line. These are the digvalayas or the azimuth circles.

Let ZRn be one of the azimuth* circles of azimuth A. Then the tangent to the circle ZRn at Z will make with Zn an angle A. If T' be the centre of the circle ZRn.

$$\widehat{QT'Z} = A$$

$$\therefore \quad QT' = QZ \cot A. \quad \text{But } QZ = r \sec \phi$$

$$\therefore \quad QT' = r \sec \phi . \cot A.$$

and since

$$T'Z^2 = T'Q^2 + QZ^2$$

$$\therefore \quad T'Z^2 = r^2 \sec^2 \phi . \cot^2 A + r^2 \sec \phi$$

$$= r^2 \sec^2 \phi . \ \operatorname{cosec}^2 A$$

$$\therefore \quad T'Z = r \sec \phi . \operatorname{cosec} A.$$

7. To determine the hour circle make twelve equal divisions of the portions of the tropics of capricorn, cancer and the equator. These should be marked (1), (2),......upto (12), in each of the circles. With (1) of the equator as centre draw a circle (α) passing through (1) of the cancer, and with (1) of the tropic of capricorn and the same radius draw another circle (β) to cut the circle (α). Then take (1) of the tropic of cancer and the same radius draw a circle (γ). Lines joining the points of intersection of (α), (β), and (α), (γ) would intersect at a point which is the centre of the first hour circle. Having obtained the centre draw a circle passing through the centres of the circles (1) of the tropics and the equator. This will be the circle of the first hour. Draw all the twelve circles in a similar way.

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* In order to avoid complications the points R, T' are not shown in the figure.

To describe regular hour circles i. e. one twenty-fourth of the day and night :—Divide the equator into twentyfour equal parts and draw twenty radiating straight lines from the pole to the points of division.

8. To describe the ecliptic :—VN`is the diameter parts of the ecliptic (see figure on page 191). And within it the circles are to be drawn exactly like the first figure and such a disk is to be taken and is to be pivoted at the pole as the ecliptic.

Divide the square of the radius of the equator by the radius of the capricorn, add the result to the radius of the tropic of capricorn and the sum gives the diameter of the ecliptic.

At 23° 27′ from the pole along the meridian is the pole of the ecliptic. One end of the thread being fixed there, the other end moving on the graduations of the equator marks the same graduations of the ecliptic wherever it cuts the ecliptic.

9. To find the position of the stars on the ecliptic.

Knowing the equatorial and ecliptic distances from the equinox, the declination and latitude of the stars and planets, we can fix the position of the stars and the planets on the disc. Count as many degrees north or south of the equator from the east point as represent the declination of the star north or south. Mark the point and join it to the south of the equator. Mark the intersections of this line and the east west line EW produced if necessary. With the pole as centre and its distance from this intersection as the radius describe, a circle. This is the declination parallel of the star. Now mark the spot indicating the ecliptic distance of the star on the ecliptic from the nearest equinox. Join the pole to this

point and the point where this line cuts the parallel of declination is the position of the star.

Otherwise draw the horizon at 66°, 33′ and the digvalayas as before. Now this horizon becomes the ecliptic and the digvalayas the great circles passing through the pole of the ecliptic. Mark off the degrees representing the celestical latitude on that perpendicular circle which passes through the graduations representing the position of the star on the ecliptic. This point is the position of the star.

Such elaborate work has been done by Jai Singh and we have tested its correctness by mathematical calculations wherever necessary. It is evident to what extent this instrument has become useful to Jai Singh and the other Mathematicians. As usual with him, Kaye has tried to show that this instrument is nothing but an astrolabe made use of by Ulukhbeg and others. It is true that such astrolabes were common in India and other Babylonian countries, but the extent to which the elaborateness and the accuracy were attained by Jai Singh's instrument were never achieved by any other astrolabes. It is uncertain that the Arabs called such an instrument an astrolabe or some other instruments of this type for, astrolabe means the instrument for measuring the altitude of the heavenly body. All the same Kaye has taken fruitless troubles to make a statement that there was nothing original in this instrument. But it must again be borne in mind that the remarkable accuracy and ease found in the *Yantra Rāj of Jai Singh is not at all found in any foreign astrolabe even if for the time being we agree to bestow upon the non-Indians the credit of originality of the invention of this instrument.

* There is one Yantrarāj instrument in Bhandarkar Research Institute at Poona.

This in itself shows the superiority of Jai Singh's genius in the field of Astronomy. It is because on the eve of the invention of the telescope and the spectroscope, the old astronomy was made perfect by the monumental masonry instruments of Jai Singh, his works are not valued nowadays, but from the historical point of view we have to place on record the gratitude of mankind to the genius of a man who worked for its betterment. So we must look upon Jai Singh's labours as the last expiring efforts of the old astronomy—as the last and the most praiseworthy attempt to probe into the mysteries of the universe by the unaided human eye.

There are other instruments in the observatories built by Jai Singh such as the Rām Yantra, the Digaṁśa Yantra or the instrument for finding the azimuth and the altitude of the heavenly bodies, the Dakṣiṇovṛtta Yantra or the meridian circle, the Ṣaṣṭāṁśa Yantra or the sextant, the nāḍivalaya Yantra &c., but they being common their construction is not given here.

WORKS CONSULTED

(1) Śulva Sūtra by Baudhāyan, Āpastamb and Kātyāyan.

(2) G. Thibaut on Śulva Sūtra Asiatic Journal of Bengal Vol. XLIV 1875.

(3) Astronomie, Astrologie und Mathematik Von G Thibaut Strassburg Verlag Von karl J. Trübner 1899.

(4) History of Indian Astronomy (in Marathi) by S. B. Dixit 1896.

(5) Positive Sciences of the Ancient Hindus by Brijendranath Seal.

(6) Alberuni's India edited with notes by E. C. Sachan Vol. I and II.

(7) Diophantus of Alexandria by T. L. Heath 1910 2nd edition.

(8) Bakṣāli Manuscript by (i) G. R. Kaye Central Publication Branch Calcutta Govt. of Bombay 1927, Vol. I and II and Vol. III 1933. (ii) Indian Antiquary 1888 by Rudolf Hoernle pages 33–48 and pages 275–279.

(9) Indian Wisdom by M. Williams.

(10) Orion by B. G. Tilak.

(11) Asiatic Researches by William Hunter.

(12) Sūryasiddhānta (i) Hand written Manuscript (Bhandarkar Oriental Research Institute, Poona) (ii) by Bapudeo Shastri of Benares edited by Lancelot Wilkinson Calcutta, the Baptist Mission Press 1861 (iii) Translation of Sūrya Siddhānta by Burgess.

(13) Āryabhaṭīyaṁ by Āryabhaṭṭa with Commentary by Parameśwarācārya.

(14) Pañca Siddhāntikā (Handwritten MS. Bhandarkar Oriental Research Institute, Poona).

(15) Brahmasphuṭasiddhānta with commentary by Pṛthudakaswāmi edited by Prof. Dvivedi.

(16) Gaṇitasāra Saṅgraha by Mahavirācārya edited by Raṅgācārya 1912.

(17) Līlāvatī, Bījagaṇita and Siddhānta Śiromaṇi by Bhāskarācārya, (i) Līlāvatī translated by U. S. Erandolkar 1889 (ii) Gaṇitādhyāya by Khānāpurkar 1913. (iii) Līlāvatī or a treatise on Arithmetic and Geometry by Taylor published in 1815. (iv) Bījagaṇita of the Hindus by Edward Strachey 1813. (v) Algebra with Arithmetic and Mensuration from the Sanskrit of Brahmagupta and Bhāskar translated by Colebrooke H. T. 1817.

(18) Astronomical Observatories of Jai Singh by G. R. Kaye.

(19) The Jaipur Observatories and its builder by Garrett R. E. 1902.

(20) Guide to the Observatories by G. R. Kaye 1920.

(21) Cajori's History of Mathematics.

(22) History of Hindu Mathematics Vol. I and II by Datta and Singh.

The following system of transliteration has been adopted in the book.

अ a, आ ā, इ i, ई ī, उ u, ऊ ū, ऋ ṛ, ए e, ऐ ai, ओ o, औ au, अं aṁ, अः ḥ, क् k, ख् kh, ग् g, घ् gh, ङ् ṅ, च् c, छ् ch, ज् j, झ् jh, ञ् ñ, ट् ṭ, ठ् ṭh, ड् ḍ, ढ् ḍh, ण् ṇ, त् t, थ् th, द् d, ध् dh, न् n, प् p, फ् ph, ब् b, भ् bh, म् m, य् y, र् r, ल् l, ळ् ḷ, व् v, श् ś, ष् sh, स् s, and ह् h.

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